Polytropes and Tropical Eigenspaces
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Background

- **Max-plus**: \((\mathbb{R}, \oplus, \odot)\). Tropical torus: \(\mathbb{T}\mathbb{P}^{n-1} = \mathbb{R}^n \setminus \langle 1, \ldots, 1 \rangle\)
- Matrix \(A \in \mathbb{R}^{n \times n}\). Eigenvalue \(\lambda(A)\).
- Eigenspace \(\text{Eig}(A)\): \(x \in \mathbb{T}\mathbb{P}^{n-1}\) such that
  \[
  A \odot x = \lambda(A) \odot x
  \]
- Polytrope \(\text{Pol}(A)\): \(x \in \mathbb{T}\mathbb{P}^{n-1}\) such that
  \[
  A \odot x \leq \lambda(A) \odot x
  \]
- \(\bar{A} = A \odot (-\lambda(A)), \bar{A}^+ = \bar{A} \oplus \bar{A}^2 \oplus \ldots, \bar{A}^* = I \oplus \bar{A}^+\).
- Extreme tropical vectors of \(\text{Pol}(A)\) = columns of \(\bar{A}^*\).
- Extreme tropical vectors of \(\text{Eig}(A)\) = common columns of \(\bar{A}^*\) and \(\bar{A}^+\).
- Identify $Pol(A), Eig(A)$ with their tropically extreme vectors.
- Piecewise-linear maps $A \mapsto \lambda(A), A \mapsto Eig(A), A \mapsto Pol(A)$ partition $\mathbb{R}^{n \times n}$ into cones of linearity.
- Common lineality space $V_n$: dimension $n$, spanned by $[e_i - e_j]$ and the all 1’s matrix.
- Goal: understand these partitions.
**Main result**

**Theorem** (Cg79, ST11, T12)

<table>
<thead>
<tr>
<th>$A \mapsto \lambda(A)$</th>
<th>$A \mapsto Eig(A)$</th>
<th>$A \mapsto Pol(A)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{R}^{n \times n} \cong V_n \times N_n$</td>
<td>$\mathbb{R}^{n \times n} \cong V_n \times P_n$</td>
<td>$\mathbb{R}^{n \times n} \cong V_n \times F_n$</td>
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<tr>
<td>Normal fan of the cycle polytope</td>
<td>Non-fan, refines $N_n$</td>
<td>Fan, refines $P_n$</td>
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<tr>
<td>Cycles on the complete digraph</td>
<td>Semi-lattice $CP[n]$ of compatible sets of connected relations</td>
<td>Lattice $CF[n]$ of complete sets of connected relations.</td>
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</table>
Let $\Psi$ maps from $\mathcal{CF}[n]$ to cones in $\mathcal{F}_n$. The same $\Psi$ maps from $\mathcal{CP}[n]$ to cones in $\mathcal{P}_n$.

Given $\mathcal{G} \in \mathcal{CF}[n]$, one can compute

- The cone $\Psi(\mathcal{G})$
- $\text{codim}(\Psi(\mathcal{G})) = \sum_{i=1}^{k} (e_i - v_i + E_i - V_i)$
- Cone intersection $\Psi(\mathcal{H}) \cap \Psi(\mathcal{G}) = \Psi(\mathcal{H} \vee \mathcal{G})$.

Enumerate cones of $\mathcal{F}_n$ using $\mathcal{CF}[n]$:

- The $f$-vector of $\mathcal{F}_3$ is $68, 207, 267, 186, 72, 14, 1$. 
Example: The cone $\mathcal{N}_2 = \mathcal{P}_2 = \mathcal{F}_2$

\[
\begin{align*}
A_{11} &= A_{22} \\
A_{22} &= \frac{A_{12} + A_{21}}{2} \\
A_{11} &= \frac{A_{12} + A_{21}}{2} \\
A_{22} &= \max\{A_{11}, \frac{A_{12} + A_{21}}{2}\} \\
A_{11} &= \max\{A_{22}, \frac{A_{12} + A_{21}}{2}\} \\
A_{12} + A_{21} &> \max\{A_{11}, A_{22}\} \\
A_{11} &= \frac{A_{12} + A_{21}}{2}
\end{align*}
\]
Example: $\mathcal{G}$ and the face lattice of $\Psi(\mathcal{G})$
Connected relations

The saturation graph $Sat(v)$ of a column $v$ of $Pol(A)$ consists of all edges $(i, j)$ such that

$$A_{ik}v_k \leq A_{ij}v_j \quad \text{for all } k \in [n].$$

This graph specifies the linear map $A \mapsto v$. A collection of such graph, one for each column of $Pol(A)$, specifies the linear maps $A \mapsto Pol(A)$.

A connected relation on $[n]$ is a weakly connected digraph with a unique strongly connected sink component.

Lemma

$G$ is a connected relation $\iff G = Sat(v)$ for some pair $(v, A)$. 
Complete sets of connected relations

So far...

- For each column \( v \) of \( Pol(A) \), \( Sat(v) \) is a connected relation.
- The linear map \( A \mapsto Pol(A) \) is specified by a ‘suitable’ set of connected relations.

**Definition.** A list \( \mathcal{G} = (G_1, \ldots, G_k) \) of connected relations is **complete** if the following are satisfied

1. Sink partition.
2. Compatible paths.
3. No conflicting triple.

Denote the set of all such lists \( \mathcal{CF}[n] \).
Proof outline

- Prove the existence of the fan $\mathcal{F}_n$ as a fan refinement of $\mathcal{N}_n$.
- Define the join operation $\vee$ on $\mathcal{CF}[n]$. Then $\mathcal{CF}[n]$ is a join semilattice generated by its join irreducibles.
- Define a semilattice homomorphism $\Psi$ from $\mathcal{CF}[n]$ to a collection of cones in $\mathbb{R}^{n \times n}$ ordered by subset inclusion.
- Show that $\Psi$ is one-to-one between the join irreducibles of $\mathcal{CF}[n]$ and the open cones of the fan $\mathcal{F}_n$. Thus $\Psi$ is a lattice anti-isomorphism between $\mathcal{CF}[n]$ and cones of $\mathcal{F}_n$. 
Connections and potential work

- Combinatorial characterization of extreme tropical vertices of tropical polyhedra: $A \odot x \leq B \odot x$. (AGG09)
- Random tropical matrices and random mappings (AP99)
- Combinatorial types of polytropes, max-plus matrix semigroup (JK08, BS, JK10)
Figure: \( P_4 \) restricted to the subspace of skew-symmetric matrices \( \wedge_2 \mathbb{R}^4 \).