Martingale coupling of cumulative hazard and exponential variables by Azéma-Yor embedding in Brownian motion

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Meeting in memory of Marc Yor
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1. Total Hazards

2. The Azéma-Yor embedding

3. Stochastic orders

4. Azéma-Yor embedding of total hazards

5. Loose Ends
• $F_0 \subseteq F_1 \subseteq F_2 \subseteq$
• $N \geq 1$ a stopping time: $(N \leq n) \in F_n$
• $h_n := P(N = n \mid F_{n-1}), n \geq 1$
• $A_n := \sum_{k=1}^{n} h_k \uparrow A_\infty = A_N = \text{total hazard}$

Key Facts:
• $A_n - 1(N \leq n)$ is an $(F_n)$-martingale
• $\rightarrow A_\infty - 1$ as $n \rightarrow \infty$ assuming $P(N < \infty) = 1$
• Limit holds in $L^p$ for $p \geq 1$
• $EA_\infty = 1$
• $EA_\infty^p < \Gamma(p + 1)$ for $p > 1$.
• $EA_\infty^p > \Gamma(p + 1)$ for $0 < p < 1$.

Example: birthday repeat time

Birthday problem with $y$ days/year.

- $Y_1, Y_2, \ldots$ independent uniform on $\{1, \ldots, y\}$
- $N := \min\{n : Y_n \in \{Y_1, \ldots, Y_{n-1}\}\}$
- $h_n := P(N = n \mid Y_1, \ldots, Y_{n-1}) = \frac{n-1}{y}1(N \geq n)$

$$A_n := h_1 + \cdots + h_n \quad \Rightarrow$$

$$A_\infty = \frac{1}{y}(0 + 1 + \cdots + (N - 1)) = \frac{N(N-1)}{2y} \approx \frac{N^2}{2y}$$

$$E[N(N-1)] = 2y \quad \Rightarrow N \text{ is of order } \sqrt{y}$$

- Simple formula for $E[N(N-1)]$ not so obvious from
  $$P(N \geq n) = \left(1 - \frac{1}{y}\right) \cdots \left(1 - \frac{n-1}{y}\right) \approx \exp\left(- \frac{n^2}{2y}\right)$$

- $P(N/\sqrt{y} \geq x) \to e^{-x^2/2}$ as $y \to \infty$, $x \geq 0$

Let $\varepsilon$ be standard exponential: $P(\varepsilon > t) = e^{-t}$, $t \geq 0$. As $y \to \infty$:

- $\sqrt{2A_\infty} \approx N/\sqrt{y} \overset{d}{\to} \sqrt{2\varepsilon}$ and $A_\infty \overset{d}{\to} \varepsilon$
Compensators

Extend to continuous time: \((\mathcal{F}_t)_{t \geq 0}, (T \leq t) \in \mathcal{F}_t\).

We know (Doob-Meyer): \(\exists!\) predictable \(\uparrow (A_t, t \geq 0)\) so that

- \(A_t - 1(T \leq t)\) is an \((\mathcal{F}_t)\)-martingale.
- \(\rightarrow A_\infty - 1 = A_T - 1\) in every \(L^p\) if \(P(T < \infty) = 1\).

Question:

- What can be said about the laws of such total hazard variables \(A_\infty\)?
Neveu’s Inequality

From Neveu *Martingales a temps discret* (1972):

\[(A_t) \text{ an } (\mathcal{F}_t)\text{-predictable ↑ process with } A_0 = 0, \ E(A_\infty) = 1.\]

\[Z_t := E[A_\infty - A_t \mid \mathcal{F}_t] = E[A_\infty \mid \mathcal{F}_t] - A_t \quad (\geq 0 \text{ super MG })\]

Suppose \(0 \leq Z \leq 1\) (bounded potential). e.g. the Azéma supermartigale

\[Z_t := P(T > t \mid \mathcal{F}_t) \text{ for some random } T \ [(\mathcal{F}_t)\text{-stopping? } \mathcal{F}_\infty\text{-meas.?}].\]

\[M_t := E[A_\infty \mid \mathcal{F}_t] = A_t + Z_t \geq 0 \quad [\text{ UI MG }]\]

Note \(M_0 = 1, \ M_\infty = A_\infty.\)

Let \(\tau_a := \inf\{t : A_t > a\}. \) Then \((A_\infty > a) = (\tau_a < \infty) \in \mathcal{F}_{\tau_a-}. \) So

\[E[A_\infty \mid A_\infty > a] = E[A_{\tau_a-} \mid A_\infty > a] + E(Z_{\tau_a-} \mid A_\infty > a)\]

\[\leq a + 1\]

\[= E(\varepsilon \mid \varepsilon > a) \text{ where } P(\varepsilon > t) = e^{-t}, \ t > 0.\]

Conclusion: \(A_\infty \leq_{\text{mrl}} \varepsilon\) and \(E(A_\infty) = E(\varepsilon) = 1\)

Distribution of \(A_\infty\) is NBUE - Barlow-Proshan(1965)

Daley 1988 - Tight bounds in exponential approximation. Mark Brown,
The Azéma-Yor Construction

\[ \psi(b) \uparrow, \quad \psi(-\infty) = 0, \quad \phi := \psi^{-1}. \]

\[ T := \inf \{ t : S_t \geq \psi(B_t) \}, \quad S_t := \sup_{0 \leq s \leq t} B_s \]

\[ P(B_T \geq b) = P(S_T \geq \psi(b)) = \exp \left( - \int_0^{\psi(b)} \frac{dy}{y - \phi(y)} \right) \]
• $\psi(b) \uparrow, \psi(-\infty) = 0, \phi := \psi^{-1}$.

• $T := \inf \{ t : S_t \geq \psi(B_t) \}$, \quad $S_t := \sup_{0 \leq s \leq t} B_s$

• $\bar{F}(b) := P(B_T \geq b) = P(S_T \geq \psi(b)) = \exp \left( - \int_0^{\psi(b)} \frac{dy}{y - \phi(y)} \right)$

• Assume $f(b) := -\bar{F}'(b)$ exists, and use $\phi(\psi(b)) = b$

\[
f(b) = \bar{F}(b) \frac{\psi'(b)}{\psi(b) - b}
\]

\[
\frac{d}{db} \left[ \bar{F}(b)\psi(b) \right] = -bf(b)
\]

\[
\bar{F}(b)\psi(b) = \int_b^\infty xf(x)dx
\]

\[
\psi(b) = \frac{\int_b^\infty xf(x)dx}{\bar{F}(b)} = E[X \mid X \geq b] \text{ for } X = B_T
\]
Azéma-Yor Theorem

For a distribution of $X$ with $E(|X|) < \infty$, define the barycenter function

$$\psi_X(b) := E(X \mid X \geq b) \quad [\quad = b \text{ if } P(X \geq b) = 0].$$

Theorem (Azéma-Yor (1979))

Let $B$ be Brownian motion. For $X$ with $E(X) = 0$ let

$$T := \inf \{ t : S_t \geq \psi_X (B_t) \} \text{ where } S_t := \sup_{0 \leq s \leq t} B_s. \text{ Then } B_T \overset{d}{=} X \text{ and } (B_{t \wedge T}, t \geq 0) \text{ is a uniformly integrable martingale}.$$

Corollary: [Dubins-Gilat (1978)]

If $(M_t)$ is a right-continous UI MG with $M_\infty \overset{d}{=} X$ then $\sup_t M_{t \leq s} S_T$. Many variations and extensions now known:

Stochastic orders


The *stochastic order*: \( X \leq_{st} Y \iff 
\begin{align*}
(i) & \quad E \phi(X) \leq E \phi(Y) \quad \forall \phi \geq 0, \uparrow; \\
(ii) & \quad P(X > a) \leq P(Y > a) \text{ for all real } a; \\
(iii) & \quad \exists X' \text{ and } Y' \text{ with } X' \overset{d}{=} X, Y' \overset{d}{=} Y \text{ and } P(X' \leq Y') = 1.
\end{align*}

The *convex order*: For integrable \( X \) and \( Y \): \( X \leq_{cx} Y \iff 
\begin{align*}
(i) & \quad E \phi(X) \leq E \phi(Y) \quad \forall \text{ convex } \phi \\
(ii) & \quad E(X) = E(Y) \text{ and } E(X - a)_+ \leq E(Y - a)_+ \text{ for all real } a \\
(iii) & \quad E(X) = E(Y) \text{ and } E|X - a| \leq E|Y - a| \text{ for all real } a \\
(iv) & \quad \exists X' \text{ and } Y' \text{ with } X' \overset{d}{=} X, Y' \overset{d}{=} Y \text{ and } E(Y' | X') = X'.
\end{align*}

The **mean residual life order**: For integrable $X$ and $Y$: $X \leq_{mrl} Y \iff$

(i) $E[X - a \mid X \geq a] \leq E[Y - a \mid Y \geq a]$ for all $a$ (with convention)

(ii) $\Psi_X(a) \leq \Psi_Y(a)$ for all $a$, for $\Psi_X(a) := E[X \mid X \geq a]$ as before.

Corollary of the Azéma-Yor embedding:

\[ X \leq_{mrl} Y \text{ and } E(X) = E(Y) \Rightarrow X \leq_{cx} Y \quad (4) \]

[Shift to $E(X) = E(Y) = 0$, then embed in BM with $T_X \leq T_Y$.]

– van der Vecht (1986), Madan-Yor (2002)

**Warning**: converse of (4) is false. Indirect argument: if true then $T_Y$ would be an *ultimate time* $T$ for the distribution of $Y$ [Meilijson 1982] meaning:

$B_T \overset{d}{=} Y$ and $\forall X$ with $X \leq_{cx} Y \exists$ stopping $S \leq T$ with $B_S \overset{d}{=} X$.

But (Meilijson and van der Vecht, 1980s): *the only ultimate times for BM are the first hitting times of \{a, b\} for some a, b.*
Azéma-Yor embedding of total hazards

Example: Birthday repeat time for $y = 4$ days/year.
Setting: \((\mathcal{F}_t)_{t \geq 0}, (T \leq t) \in \mathcal{F}_t, P(T < \infty) = 1\)

- \(A_t - 1(T \leq t)\) is an \((\mathcal{F}_t)\)-martingale
- \(\rightarrow A_\infty - 1\).
- Assume a uniform \([0, 1]\) variable \(U\) independent of \(\mathcal{F}_\infty\).

**Theorem**

*There exists a standard exponential variable \(\varepsilon\) such that*

\[
E(\varepsilon | \mathcal{F}_\infty) = A_\infty \tag{5}
\]

\[
E[(\varepsilon - A_\infty)^2 | \mathcal{F}_\infty] = \Delta A_T := A_T - A_{T^-} \tag{6}
\]

\[
E[(\varepsilon - A_\infty)^2 ] = E[\Delta A_T] = E \sum_s (\Delta A_s)^2 \tag{7}
\]

**Remarks**

- (5) follows from Neveu’s inequality that \(A_\infty \leq_{\text{mrl}} \varepsilon\).
- (6) involves details of the Azéma-Yor embedding.
Details of the coupling

For each $t > 0$ there is a unique $p = (1 - e^{-t})/t \in (0, 1)$ so

$$\xi(t) \overset{d}{=} p \text{Dist}(\varepsilon | \varepsilon < t) + (1 - p)\text{Dist}(\varepsilon | \varepsilon > t)$$

has

$$E[\xi(t)] = pE(\varepsilon | \varepsilon < t) + (1 - p)E(\varepsilon | \varepsilon > t) = t$$

Also $\text{Var}(\xi(t)) = t$. Explicitly, take $U, V, \varepsilon'$ are independent, with $U, V$ uniform$[0, 1]$ and $\varepsilon' \overset{d}{=} \varepsilon$, and set

$$\xi(t) = tU 1(V \leq e^{-tU}) + (t + \varepsilon')1(V > e^{-tU}).$$

For $A_T$ a total hazard, take $U, V, \varepsilon'$ indpt. of $(A_T-, A_T)$. Let

$$\varepsilon = A_T- + \xi(A_T - A_T-)$$

so

$$E(\varepsilon | A_T-, A_T) = A_T$$

and

$$E[(\varepsilon - A_T)^2 | A_T-, A_T) = A_T - A_T-.$$
Exponential coupling

Example: discrete distribution of $X$ with constant hazards

$$h^*_n = P(X = n \mid X \geq n) \text{ and } A^*_n := \sum_{i=1}^{n} h^*_i$$
Characterize all possible laws of total hazard variables $A_T$. (Know extremes. Simplex?)

Can show $\gamma_r/r$ is $\downarrow$ in MRL as $r \uparrow$. (So Madan-Yor $\Rightarrow$ reverse peacock). $\gamma_1 \overset{d}\equiv \varepsilon$. Is $\gamma_r/r$ a total hazard for $r > 1$?

What about $\gamma_r - r$. Is this $\uparrow$ in MRL?

Embedding the entire martingale $A_t - 1(T \leq t)$ in BM.

What about the non-adapted case (martingale derived from a potential)?

Suppose a stopping time $S \leq T_Y$ where $T_Y$ is the Azéma-Yor time for embedding $Y$ in BM. Does that imply $B_{S \leq \text{mrl}} Y$?