1. Let $x_t = e^{2\pi i \omega t}$ for some $\omega \in (-\frac{1}{2}, \frac{1}{2}]$. Show that

$$d(\omega_j) = n^{-\frac{1}{2}} e^{\pi i (\omega - \omega_j)(n-1)} D_n(\omega - \omega_j),$$

where $D_n(\omega) = \frac{\sin(n\pi(\omega-\omega_j))}{\sin(\pi(\omega-\omega_j))}$ is the Dirichlet kernel. Plot the Dirichlet kernel, and explain what happens to the DFT coefficients if the Fourier frequencies do not exactly match the frequency of the signal.

2. Consider the following ARMA($p,q$) processes, all of the form $\phi(B)x_t = \theta(B)w_t$. Calculate the spectral density. Sketch the location of the poles and zeros in the complex plane, and describe how they affect the spectral density.

- $\phi(z) = 1 + (\frac{9}{10}z)^2$, $\theta(z) = 1 + \frac{1}{3}z$,
- $\phi(z) = 1 - 2z + 2z^2$, $\theta(z) = 1 - \frac{1}{2}z$,
- $\phi(z) = 1 - 4z^2$, $\theta(z) = 1 - z + z^2$,
- $\phi(z) = 1 + \frac{3}{2}z$, $\theta(z) = 1 + \frac{1}{4}z^2$.

3. Generate four sample paths of length $n = 128, 256, 512, 1024$ of the AR(1) process $\phi(B)x_t = w_t$, where $\phi(z) = 1 - \frac{1}{2}z$. Plot the periodogram for each sample path. Calculate an approximate 95% confidence interval for $f(0.1)$. Does the confidence interval shrink as $n$ grows?

4. Consider the smoothed periodogram:

$$\hat{f}(\omega) = \frac{1}{2\lceil \sqrt{n} \rceil + 1} \sum_{|j| < \sqrt{n}} I(\omega(j) + \frac{j}{n}),$$

where $I$ is the periodogram and $\omega(j)$ is the Fourier frequency closest to $\omega$. Plot the smoothed periodogram for each of the sample paths in Q3.