Problem 1

We evaluate \(d(\omega_j)\) by a direct calculation,

\[
d(\omega_j) = n^{-\frac{1}{2}} \sum_{t=0}^{n-1} x_t e^{-2\pi i \omega_j t} = n^{-\frac{1}{2}} \sum_{t=0}^{n-1} e^{2\pi i (\omega - \omega_j) t}
\]

\[
= \frac{1}{\sqrt{n}} \left[ 1 - e^{2\pi i (\omega - \omega_j) n} \right] \frac{e^{\pi i (\omega - \omega_j) n} - e^{\pi i (\omega - \omega_j) n}}{e^{\pi i (\omega - \omega_j) n} - e^{\pi i (\omega - \omega_j)}}
\]

Since \(e^{-i\theta} - e^{i\theta} = -2i \sin \theta\), the third term in the product is

\[
\frac{e^{-\pi i (\omega - \omega_j) n} - e^{\pi i (\omega - \omega_j) n}}{e^{-\pi i (\omega - \omega_j)} - e^{\pi i (\omega - \omega_j)}} = \frac{\sin(\pi i (\omega - \omega_j) n)}{\sin(\pi i (\omega - \omega_j))},
\]

which is exactly the Dirichlet kernel.

Problem 2

(a)

The AR polynomial is \(\phi(z) = 1 + \left(\frac{9}{10}\right)z^2\), which has roots (zeros) \(z = \pm \frac{10i}{9}\). The MA polynomial is \(\theta(z) = 1 + z/3\), which has a root (pole) \(z = -3\). The spectral density is

\[
f_X(v) = \left| \frac{1 + \frac{1}{3}e^{2\pi iv}}{1 - \left(\frac{9}{10}\right)^2e^{4\pi iv}} \right|^2
\]

The spectral density will increase whenever \(e^{2\pi iv} = -1\), i.e. \(v = \frac{1}{2}\). The spectral density will decrease whenever \(e^{2\pi iv}\) is close to the zeros \(\pm \frac{10i}{9}\), and \(f_X(v)\) will reach the minimum when \(e^{2\pi iv} = \pm i\), i.e. \(v = \frac{1}{4}\).

(b)

The AR polynomial is \(\phi(z) = 1 - 2z + 2z^2\), which has roots (zeros) \(\frac{1}{2} \pm \frac{1}{2}i\). The MA polynomial is \(\theta(z) = 1 - \frac{1}{2}z\), which has a root (pole) 2. The spectral density is

\[
f_X(v) = \left| \frac{1 - \frac{1}{2}e^{2\pi iv}}{1 - 2e^{2\pi iv} + 2e^{4\pi iv}} \right|^2
\]

The spectral density will increase whenever \(e^{2\pi iv}\) is close to the pole 2, and \(f_X(v)\) will reach the maximum when \(e^{2\pi iv} = 1\), i.e. \(v = 0\). The spectral density will decrease whenever \(e^{2\pi iv}\) is close to the zeros \(\frac{1}{2} \pm \frac{1}{2}i\), and \(f_X(v)\) will reach the minimum when \(e^{2\pi iv} = \frac{\sqrt{2}}{2} \pm \frac{\sqrt{2}}{2}i\), i.e. \(v = \frac{1}{8}\).
(c) The AR polynomial is $\phi(z) = 1 - 4z^2$, which has roots (zeros) $\pm 1/2$. The MA polynomial is $\theta(z) = 1 - z + z^2/2$, which has roots (poles) $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. The spectral density is

$$f_X(v) = \left| \frac{1 - e^{2\pi iv} + e^{4\pi iv}}{1 - 4e^{4\pi iv}} \right|^2$$

The spectral density will increase whenever $e^{2\pi iv}$ is close to the pole $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, and $f_X(v)$ will reach the maximum when $e^{2\pi iv} = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, i.e. $v = \frac{1}{3}$. The spectral density will decreases whenever $e^{2\pi iv}$ is close to the zeros $\frac{1}{2} \pm \frac{1}{2}i$, and $f_X(v)$ will reach the minimum when $e^{2\pi iv} = 1$, i.e. $v = 0$.

(d) The AR polynomial is $\phi(z) = 1 + 3z/4$, which has a root (zero) $-4/3$. The MA polynomial is $\theta(z) = 1 + z^2/9$, which has roots (poles) $\pm 3i$. The spectral density is

$$f_X(v) = \left| \frac{1 + \frac{1}{3}e^{4\pi iv}}{1 + \frac{3}{4}e^{2\pi iv}} \right|^2$$

The spectral density will increase whenever $e^{2\pi iv}$ is close to the pole $\pm 3i$, and $f_X(v)$ will reach the maximum when $e^{2\pi iv} = i$, i.e. $v = \frac{1}{4}$. The spectral density will decreases whenever $e^{2\pi iv}$ is close to the zeros $-4/3$, and $f_X(v)$ will reach the minimum when $e^{2\pi iv} = -1$, i.e. $v = \frac{1}{2}$.

Problem 3

See Figure 1 for a plot of the periodograms. The code that generated them (and computed the confidence intervals) was:

```r
set.seed(153)
psi <- function(z) 1/(1 - 0.5*z)
f <- function(x) abs(psi(exp(2i*pi*x)))^2
plotPGram <- function(data, smooth) {
  k <- kernel("daniell", if (smooth) floor(sqrt(length(data))) else 0)
title <- sprintf("Raw periodogram for %d samples", length(data))
  if (smooth)
    title <- sprintf("Smoothed periodogram for %d samples", length(data))
p <- spec.pgram(data, k, taper=0, log="no", ylim=c(0,20), main=title)
grid <- (0:50) / 100
lines(grid, f(grid), lty=2)
df <- p$df
U <- df / qchisq(0.025, df)
L <- df / qchisq(0.975, df)
len <- length(p$spec)
idx <- round(len/5)
c(p$spec[idx] * L, p$spec[idx] * U)
```
```r
Q3.plot <- function(n) {
  x <- arima.sim(model=list(ar=0.5), n)
  plotPGram(x, F)
}
par(mfrow=c(2,2))
Q3.plot(128); Q3.plot(512); Q3.plot(1024); Q3.plot(2048)
```

Our confidence intervals for the four simulations were

\[ 0.07300497, 10.63704035 \], \[ 0.01879657, 2.73871585 \], \[ 1.713544, 249.668451 \] and \[ 0.9218415, 134.3150317 \]. Clearly, we are not very confident in the unsmoothed periodogram, even for large samples sizes. This is consistent with the asymptotic theory, which says that each point of the periodogram has a non-zero asymptotic variance.

### Problem 4

See Figure 2 for a plot of the periodograms. The code that generated them was (in addition to the code in the previous question):

```r
set.seed(153)
Q4.plot <- function(n) {
  x <- arima.sim(model=list(ar=0.5), n)
  plotPGram(x, T)
}
par(mfrow=c(2,2))
Q4.plot(128); Q4.plot(512); Q4.plot(1024); Q4.plot(2048)
```
Figure 1: Unsmoothed periodograms at different sample sizes.
Figure 2: Smoothed periodograms at different sample sizes.