1. **Testing white noise:** Consider testing the null hypothesis that a time series is white noise. Is the test that rejects when any of \( \sqrt{n} |\hat{\rho}(1)|, \ldots, \sqrt{n} |\hat{\rho}(h)| \) exceeds the \( 1 - \frac{\alpha}{2} \) quantile of the standard normal distribution a valid test of the null hypothesis? That is, is the Type 1 error rate (asymptotically) at most \( \alpha \)? If not, suggest a valid test of the aforementioned null.

2. **Autocovariance of a linear process:** Let \( x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \), where \( \{\psi_j\} \) is absolutely summable. Show that its autocovariance function is \( \gamma(h) = \sigma_w^2 \sum_{j=0}^{\infty} \psi_j + h \psi_j \).

You may assume the series converges.

3. **MA(\( \infty \)):** Let \( x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \). Shows that as long as \( \sum_{j=0}^{\infty} |\psi_j| < \infty \), the sum in the definition of \( x_t \) converges in mean square.

4. **Matching coefficients for an ARMA(p,q) process:** Consider the following ARMA(p,q) processes, all of the form \( \phi(B)x_t = \theta(B)w_t \),

- \( \phi(z) = 1 + (\frac{9}{10})^2 z, \theta(z) = 1 + \frac{1}{3} z \),
- \( \phi(z) = 1 - 3z, \theta(z) = 1 + 2z - 8z^2 \),
- \( \phi(z) = 1 - 4z^2, \theta(z) = 1 - z + \frac{1}{7} z^2 \),
- \( \phi(z) = 1 - \frac{9}{4} z - \frac{9}{4} z^2, \theta(z) = 1 - 3z + \frac{1}{3} z^2 - \frac{1}{3} z^3 \).

Determine whether they are causal and/or invertible.

5. **Homogeneous linear difference equations:** Consider a \( p \)-th order homogeneous linear difference equation \( \phi(B)x_t = 0 \), where

\[ \phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p. \]

Let \( \lambda_k \) be the distinct roots of \( \phi(z) \) and \( m_k \) be the multiplicity of the \( k \)-th root (so \( \sum_k m_k = p \)). The general solution is given by

\[ x_t = \sum_k \lambda_k^{t-1} (\sum_{l=0}^{m_k-1} c_{k,l} t^l), \]

where the coefficients \( c_{k,l} \) are determined by the initial conditions.
• Assume there are no multiple roots \((m_k = 1)\). Simplify the expression of the general solution and check that it solves the difference equation. If you’re feeling brave, check that the general solution solves the difference equation (allowing the possibility of multiple roots).

• Let \(\phi(z) = 1 - z + \frac{1}{3}z^2\). What is the general solution to the associated homogeneous linear difference equation? What is the particular solution that obeys the initial conditions \(x_0 = 1, x_1 = 0\)?