1 Fourier series

Fourier series, usually taught in a first course on calculus, concerns the approximation of periodic functions on $\mathbb{R}$ by trigonometric polynomials. Since such functions are completely determined by their restriction to a single period, we consider the approximation of functions on an interval by trigonometric polynomials.

Without loss of generality, we take the interval to be $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Recall the set of square integrable functions on $\left(-\frac{1}{2}, \frac{1}{2}\right)$:

$$L^2\left(-\frac{1}{2}, \frac{1}{2}\right) = \left\{ f : \left(-\frac{1}{2}, \frac{1}{2}\right) \to \mathbb{C} : \int_{-1/2}^{1/2} |f(x)|^2 \, dx < \infty \right\}.$$ 

It is a Hilbert space if equipped with the inner product

$$\langle f, g \rangle = \int_{-1/2}^{1/2} f(x) \overline{g(x)} \, dx.$$ 

We seek to approximate any $f \in L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ by a linear combination of complex exponentials:

$$e_j(x) = e^{(2\pi i)jx}, \quad x \in \left(-\frac{1}{2}, \frac{1}{2}\right).$$

Such linear combinations are called trigonometric polynomials because

$$e^{(2\pi i)jx} = (e^{2\pi ix})^j = (\cos(2\pi x) + i \sin(2\pi x))^j.$$ 

Thus linear combinations of complex exponentials are polynomials of sinusoids.

**Theorem 1.1.** The set of complex exponentials $\{e_j\}_{j=-\infty}^{\infty}$ is a complete orthonormal sequence in $L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$. That is,

1. $\{e_j\}_{j=-\infty}^{\infty}$ is complete: the only function $f \in L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ that is orthogonal to all the $e_j$’s is the zero function.

2. $\{e_j\}_{j=-\infty}^{\infty}$ is orthonormal: $\langle e_j, e_k \rangle$ is 0 if $j \neq k$ and 1 if $j = k$.

**Proof.** We check that $\{e_j\}_{j=-\infty}^{\infty}$ is orthonormal:

$$\langle e_j, e_k \rangle = \int_{-1/2}^{1/2} e_j(x) \overline{e_k(x)} \, dx = \int_{-1/2}^{1/2} e^{2\pi i(j-k)x} \, dx.$$
If $j \neq k$, evaluating the integral gives

$$= \frac{1}{2\pi i(j - k)} (e^{\pi i(j - k)} - e^{-\pi i(j - k)})$$

$$= \frac{\sin(\pi i(j - k))}{\pi(j - k)} = 0.$$ 

If $j = k$, the integrand is 1. Thus the integral is $\int_{-1/2}^{1/2} dx = 1$. \hfill \Box

The second property, orthonormality, should be familiar. The first property, completeness, is interpretable as \{e_j\}_{j=-\infty}^{\infty} spans $L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$. Complete orthonormal sequences are comparable to orthonormal bases. Any function $f \in L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ can be expressed as a linear combination of the $e_j$'s:

$$f(x) = \sum_{j=-\infty}^{\infty} \hat{f}_j e_j(x). \quad (1.1)$$

Further, the coefficients $\hat{f}_j$ are given by

$$\langle f, e_j \rangle = \int_{-1/2}^{1/2} f(x) e_j(x) dx = \int_{-1/2}^{1/2} f(x) e^{-(2\pi i)j x} dx, \quad (1.2)$$

which are exactly the Fourier coefficients of $f$. Since it is possible to reconstruct any $f \in L^2\left(-\frac{1}{2}, \frac{1}{2}\right)$ given its Fourier coefficients \{\langle f, e_j \rangle\}_{j=-\infty}^{\infty}, the sequence of Fourier coefficients is an alternative representation of the function. Another important consequence of the completeness and orthonormality of the $e_j$'s is Parseval’s formula:

$$\|f\|^2 = \sum_{j=-\infty}^{\infty} |\langle f, e_j \rangle|^2. \quad (1.3)$$

To gain intuition, consider the analogous statements to (1.1), (1.2), and (1.3) in $\mathbb{C}^n$. We know the Fourier basis \{f_j\}_{j=0}^{n-1}, where

$$f_j = [e^{(2\pi i)\omega_0} \ldots e^{(2\pi i)\omega_{n-1}(n-1)}]^T \text{ and } \omega_j = \frac{j}{n}$$

1Checking \{e_j\}_{j=-\infty}^{\infty} is complete is technical. We skip the details here.
is an orthonormal basis of $\mathbb{C}^n$. Thus it is possible to express any $x \in \mathbb{C}^n$ as a linear combination of $f_j$’s:

$$x = \sum_{j=0}^{n-1} \hat{x}_j f_j,$$

which is analogous to (1.1). In matrix form, we have $x = F \hat{x}$, where

$$F = \begin{bmatrix} | & | & | & | \\ e_0 & \ldots & e_{n-1} & | \\ | & | & | & | \\ \end{bmatrix} \quad \text{and} \quad \hat{x} = \begin{bmatrix} \hat{x}_0 \\ \vdots \\ \hat{x}_{n-1} \end{bmatrix}.$$

We solve for $\hat{x}$ to obtain $\hat{x} = F^* x$. Thus the $j$-th coefficient is given by

$$\hat{x}_j = x^* e_j = \langle x, e_j \rangle,$$

which is analogous to (1.2). Finally, the squared norms of $x$ and $\hat{x}$ are equal:

$$\|\hat{x}\|^2 = \hat{x}^* \hat{x} = (F^* x)^*(F^* x) = x^* (F F^*) x = x^* x = \|x\|^2,$$

which is analogous to (1.3). The fourth equality is a consequence of the fact that the $f_j$’s span $\mathbb{C}^n$ and that they are orthonormal.

There is a connection between the smoothness of a function and the decay of its Fourier coefficients. Intuitively, if a function is smooth, its Fourier coefficients associated with higher frequency complex exponentials should be small. Since the function is smooth, it is well-approximated by a linear combination of slowly-varying sinusoids. Thus the Fourier coefficients of a smooth function decay rapidly. Conversely, the Fourier coefficients of a non-smooth function decays slowly. We state a theorem that formalizes this connection.

**Theorem 1.2.** If $f : (-\frac{1}{2}, \frac{1}{2})$ is $k$-times continuously differentiable,

$$\langle f, e_j \rangle \lesssim (2\pi ij)^{-k} \|f^{(k)}\|,$$

where $\|f\| = \langle f, f \rangle^{1/2}$.

The proof of Theorem 1.2 hinges on a crucial fact:

$$\langle f', e_j \rangle = (2\pi ij) \langle f, e_j \rangle.$$

(1.4)
To derive the preceding identity, we integrate by parts (and drop any vanishing terms):

\[ \langle f', e_j \rangle = \int_{-1/2}^{1/2} f'(x)e^{-(2\pi i)jx} \, dx \]

\[ = -\int_{-1/2}^{1/2} f(x)(-2\pi i j)e^{-(2\pi i)jx} \, dx \]

\[ = (2\pi i j)\langle f, e_j \rangle. \]

Generally speaking, the derivative of a function is “rougher” than the function: the derivative of a \( k \)-times continuously differentiable function is only \( k-1 \)-times continuously differentiable. Thus the Fourier coefficients of \( f' \) should decay slower than those of \( f \). The identity (1.4) agrees with our intuition: if the Fourier coefficients of \( f \) decay at the rate \( O(j^{-k}) \), the Fourier coefficients of \( f' \) decay at the slower rate \( O(j^{-(k-1)}) \).

**Proof.** Let \( f^{(k)} \) be the \( k \)-th derivative of \( f \). The Fourier coefficients of \( f \) are

\[ \langle f, e_j \rangle = (2\pi i j)^{-k} \langle f^{(k)}, e_j \rangle, \]

where we applied (1.4) \( k \)-times. By the Cauchy-Schwartz inequality,

\[ \langle f^{(k)}, e_j \rangle \leq \| f^{(k)} \| \| e_j \|. \]

We recall \( \| e_j \| = 1 \) to obtain the stated result. \( \square \)