p-adic Harmonic analysis

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Abstract. In this expository paper, we present the theory of Harmonic analysis on the field of p-adic numbers . These are locally compact, non-discrete, totally disconnected topological fields. Hence, this harmonic analysis is different in flavor compared to that on euclidean spaces.

Analyse Harmonique p-adique

Résumé. Ces notes ont pour but de présenter la théorie de l'analyse harmonique p-adique (et en general sur les corps locaux). Ce derniers sont des corps topologique localement compacts, non-discrets et totalement discontinu. La structure topologique non intuitive de ces corps fait que cette théorie est nettement différent de la théorie classique.

1. Introduction and background on local fields

A local field is a locally compact, non-discrete, totally disconnected topological field. They have been introduced by Kurt Hensel in the 19th century and were originally studied in number theory [Cas86]. However, local fields have found a wide spectrum of applications from the study of error-free computation [GK12] to mathematical physics [Khr90]. Moreover, they have become objects of interest in their own right. There is an extensive literature on valued fields in number Theory [Ser13, Wei13, EP05], analysis [vR78, Sch84, Sch07], representation theory [CR66], mathematical physics [VVZ94, Khr13], and probability [Eva01, EL07, AZ01].

Two fundamental examples of local fields are the field of p-adic numbers $\mathbb{Q}_p$ and the field $\mathbb{F}_q((t))$ of formal Laurent series in one variable t with coefficients

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in a finite field $\mathbb{F}_q$. Actually, any local field is a finite algebraic extension of either $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$.

For the remaining of this paper, we focus on the case of p-adic fields $\mathbb{Q}_p$. Much of the theory developed in this case can be easily generalized for local fields.

Let $p > 0$ be a positive prime in $\mathbb{Z}$. We can factor any rational number $r \in \mathbb{Q} \setminus \{0\}$ uniquely as $r = p^s(a/b)$ where $a, b \in \mathbb{Z}$ are not divisible by p. The number $s \in \mathbb{Z}$ is call the p-adic valuation of r and we write $s = v_p(r)$. When $r = 0$ we take $s = \infty$ by convention. We then get a map $v_p : \mathbb{Q} \to \mathbb{Z}$ satisfying the following properties:

(i) $v_p : \mathbb{Q}^\times \to \mathbb{Z}$ is a group morphism.
(ii) $v_p(x) = \infty$ if an only if $x = 0$.

The valuation map v_p defines an absolute value $|\cdot|_p$ on $\mathbb{Q}$ by taking $|x|_p = p^{-v_p(x)}$ for any $x \in \mathbb{Q}$. Moreover this absolute value is an ultrametric absolute value, i.e, it satisfies the following conditions:

$$|x|_p = 0 \iff x = 0. \tag{1.1}$$
$$|xy|_p = |x|_p|y|_=p. \tag{1.2}$$
$$|x + y|_p \leqslant \max(|x|_p, |y|_p). \tag{1.3}$$

The absolute value $|\cdot|_p$ is also called a exponential valuation and once can easily check that it's non-archimedian, i.e, $\{|n|_p, n \in \mathbb{Z}\}$ is bounded. This defines a metric on $\mathbb{Q}$ associated to the prime p.

Just like $\mathbb{R}$ is the completion of $\mathbb{Q}$ with respect to the usual absolute value $|\cdot|_\infty$, the field $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with the absolute value $|\cdot|_p$. Let $\mathbb{Z}_p := \{x \in \mathbb{Q}, |x|_p \leqslant 1\}$ the ball of radius 1. Thanks to property 1.3, the set $\mathbb{Z}_p$ is a ring. It is called the ring of p-adic integers and the field $\mathbb{Q}_p$ is its field of fractions. So we can think of the pair $(\mathbb{Z}_p, \mathbb{Q}_p)$ as the p-adic counter part of $(\mathbb{Z}, \mathbb{Q})$.

From an algebraic point of view, we can first construct $\mathbb{Z}_p$ as a projective limit of the p-adic filtration $(\mathbb{Z}/p^n\mathbb{Z})_{n\geqslant 1}$, i.e,

$$\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}.$$

Then we can construct $\mathbb{Q}_p$ as the field of fractions of $\mathbb{Z}_p$. The ring $\mathbb{Z}_p$ has a unique maximal ideal $\mathfrak{p} := \{x \in \mathbb{Q}_p, |x|_p < 1\} = p\mathbb{Z}_p$. The quotient $\mathbb{Z}_p/\mathfrak{p} = \mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$

is called the *residue field* . This p-adic filtration construction allows us to see elements of $\mathbb{Q}_p$ as power series in p with coefficient in $\{0, 1, \dots, p-1\}$, i.e,

$$\mathbb{Q}_p := \left\{ x := \sum_{n \geqslant m} a_n p^n, a_m \neq 0 \text{ and } m = v_p(x) \in \mathbb{Z} \right\}. \tag{1.4}$$

The convergence of such a series is due to the fact that the partial sums form a Cauchy sequence thanks to property 1.3 (see 2.1). We can then picture $\mathbb{Q}_p$ as an infinite tree with valence $p+1$.

Notice that, since the absolute value takes values in $\{p^n, n \in \mathbb{Z}\}$, the ball unit $\mathbb{Z}_p$ is both closed and open in the topology induced by $|\cdot|_p$. This is because we can write $\mathbb{Z}_p = \{x \in \mathbb{Q}_p, |x|_p \leqslant p\}$. We can then $\mathbb{Q}_p$ is a totally disconnected field.

In Section 2, we discuss some preliminary algebraic and topological properties of $\mathbb{Q}_p$. Section 3 will treat general facts on Fourier theory on loacally compact abelian groups while Section 4 will be devoted to additive character theory on $\mathbb{Q}_p$. Finally, we shall Fourier analysis on $(\mathbb{Q}_p, +)$ and compute some examples.

We stress that this theory, even though quite young compared to its Euclidean counterpart, is pretty well established. The author claims no originality.

2. Preliminary properties of the p-adic completion $\mathbb{Q}_p$

We begin by recalling that the ultrametric property 1.3 that the absolue value $|\cdot|_p$ enjoys, endows the space $\mathbb{Q}_p$ with a rather unintuitive topology. For example, if two balls B_1, B_2 in $\mathbb{Q}_p$ have a non empty intersection then either $B_1 \subset B_2$ or $B_2 \subset B_1$. So in this section we will start by presenting a few interesting properties of $\mathbb{Q}_p$ that will be useful in developing Fourier theory.

2.1. Additive structure

For any $n \in \mathbb{Z}$, the fractional ideal $\mathfrak{p}^n$ is a compact additive subgroup of $\mathbb{Q}_p$ and we can write

$$\mathbb{Q}_p = \bigcup_{n \in \mathbb{Z}} \mathfrak{p}^n = \bigcup_{n \in \mathbb{Z}} p^n \mathbb{Z}_p.$$

The nested ideals $\mathbb{Z}_p \supset \mathfrak{p} \supset \mathfrak{p}^2 \supset \dots (0)$ forms a basis of neighborhoods of 0. For a couple of integers $n \geqslant m$, we have $[\mathfrak{p}^{\mathfrak{m}} : \mathfrak{p}^n] = p^{n-m}$. One can think of the collection of balls in $\mathbb{Z}_p$ as being arranged in an infinite rooted p-ary tree: the root is $\mathbb{Z}_p$, and the nodes at level k are all the balls of radius p^{-k} (additive cosets of $p^k\mathbb{Z}_p$). Let's now describe open sets of $\mathbb{Q}_p$.

PROPOSITION 2.2. — *Any open set U in $\mathbb{Q}_p$ is the union of disjoint costs of $\mathfrak{p}^n$, i.e, there exists a sequence $(a_i)_{i\in I}$ of elements in U and integers $(n_i)_{i\in I}$ such that*

$$U = \bigcup_{i\in I}(a_i + \mathfrak{p}^{\mathfrak{n}_\mathfrak{i}})$$

and $(a_i + \mathfrak{p}^{\mathfrak{n}_\mathfrak{i}}) \cap (a_j + \mathfrak{p}^{\mathfrak{n}_\mathfrak{j}}) = \emptyset$ whenever $i \neq j$.

Proof. — Let $a_0 \in U$ an element of U. Since U is open there exists $n \in \mathbb{Z}$ such that $a_0 + \mathfrak{p}^n \subset U$. But the set $a_0 + \mathfrak{p}^n$ is both closed and open so the set $U \setminus (a_0 + \mathfrak{p}^n)$ is open. Choosing an element $a_1 \in U \setminus (a_0 + \mathfrak{p}^n)$ we can repeat the same argument. So the result follows by induction. □

EXAMPLE 2.3. — *The ring of integers $\mathbb{Z}_p$ is an open set. It is the disjoint union of p cosets of $\mathfrak{p}$. In general it is the disjoint union of p^n cosets of $\mathfrak{p}^n$*

$$\mathbb{Z}_p = \bigcup_{0\leqslant a_0,\dots,a_n\leqslant p-1}(a_0 + a_1 p + \cdots + a_{n-}p^{n-1} + \mathfrak{p}^n).$$

The units $\mathbb{Z}_p^\times$ of $\mathbb{Z}_p$ are exactly the elements of valuation 0. This means that $\mathbb{Z}_p^\times = \{x \in \mathbb{Z}_p, |x|_p = 1\}$. This multiplicative group is an open set and we have

$$\mathbb{Z}_p^\times = \bigcup_{1\leqslant a_0\leqslant p-1}(a_0 + \mathfrak{p}).$$

We shall revisit this multiplicative group a little bit further in the context of multiplicative structure.

Next we give a result on the converge of series in $\mathbb{Q}_p$ which is far simpler than its classical version on $\mathbb{R}$ or $\mathbb{C}$.

LEMMA 2.1. — *Let $(x_n)_{n\geqslant 0}$ be a sequence in $\mathbb{Q}_p$. Then the series $\sum_n a_n$ is convergent if and only if $a_n \xrightarrow[n\to\infty]{} 0$.*

Proof. — The only if direction is obvious. The if direction follows from 1.3 because, when $a_n \xrightarrow[n\to\infty]{} 0$, the sequence of partial sums $\sum_{k=0}^n a_k$ is Cauchy. So since $\mathbb{Q}_p$ is by definition complete, the series converges. □

This justifies the convergence of the p-adic expansion of elements of $\mathbb{Q}_p$ we have seen in (1.4). Let's now define an ingredient we will be using later in the discussion of additive characters on $\mathbb{Q}_p$.

DEFINITION 2.4. — *We define the tail of an element* $x = \sum_{n=v_p(x)} a_n p^n \in \mathbb{Q}_p$ *as follows:*

$$\omega(x) := \begin{cases} \sum_{n=v_p(x)}^{-1} a_n p^n, & \text{if } v_p(x) < 0, \\ 0, & \text{if } v_p(x) \geqslant 0 \end{cases}$$

Notice that $x \in \mathbb{Z}_p$ if and only if $\omega(x) = 0$ and that $\omega : \mathbb{Q}_p \to \mathbb{Q}$.

2.5. Multiplicative structure of $\mathbb{Q}_p$

Since we will discuss Fourier theory on the multiplicative group $\mathbb{Q}_p^\times$, we need to better understand its structure. Let us define the nested sequence of groups $U_0 \supset U_1 \supset U_2 \supset \dots$ as follows:

$$U_0 = \mathbb{Z}_p^\times \quad \text{and} \quad U_n = 1 + \mathfrak{p}^n.$$

These groups form a neighborhood basis of the unit 1. Notice that $U_0/U_1 \simeq \mathbb{F}_p^\times$ and $U_n/U_{n+1} \simeq \mathbb{F}_p$ for $n \geqslant 1$. Hence we have $[U_0, U_n] = (p-1)p^{n-1}$ for $n \geqslant 1$. The group $\mathbb{Q}_p^\times$ has a pleasant multiplicative structure since it can be decomposed as follows:

$$\mathbb{Q}_p^\times \simeq p^{\mathbb{Z}} \times U_0 \simeq \mathbb{Z} \times U_0.$$

We can also get a finer decomposition by decomposing U_0 and we obtain $\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{F}_p^\times \times U_1$. In general for $n \geqslant 1$ we have

$$\mathbb{Q}_p^\times \simeq \mathbb{Z} \times \mathbb{F}_p^\times \times \mathbb{F}_p^{n-1} \times U_n.$$

Next we establish a interesting fact concerning the field $\mathbb{Q}_p$.

PROPOSITION 2.6. — *(Teichmuller representatives) The field* $\mathbb{Q}_p$ *contains the* $(p-1)^{th}$ *roots of the unit.*

This means that we can lift the elements of the cyclic multiplicative group $\mathbb{F}_p^\times$ to $\mathbb{Z}_p$. To show this fact one can use an algebraic result called Hensel's lemma which allows to lift a factorization of polynomials, or use the following analytic argument.

Proof of Proposition 2.6. — For $\epsilon \in \mathbb{F}_p^\times = U_0/U_1$ and $x \in \mathbb{Z}_p$ such that $\epsilon = x \mod \mathfrak{p}$. Then consider the sequence $(x^{p^n})_{n\geqslant 0}$. We can show that this sequence is a Cauchy sequence in $\mathbb{Q}_p$ because $x^{p^{n+1}} = x^{p^n} \mod \mathfrak{p}^n$. Then the sequence x^{p^n} converges to an element x_ϵ. Moreover, we have $x_\epsilon^p = x_\epsilon$ and $x_\epsilon \neq 0$. Hence we deduce that $x_\epsilon^{p-1} = 1$. □

We then have a cyclic multiplicative group $\mu_{p-1} := \{x \in \mathbb{Q}_p, x^{p-1} = 1\} = \{x_\epsilon, \epsilon \in \mathbb{F}_p^\times\}$ in $\mathbb{Z}_p^\times$. The elements of this group are call Teichmuller representatives. We also have

$$\mathbb{Z}_p = \mathfrak{p} \cup \bigcup_{u \in \mu_{p-1}} (u + \mathfrak{p}).$$

3. Characters theory of locally compact abelian groups

In this section we recall some Fourier analysis facts on locally compact abelian groups which will be useful in discussing Fourier theory on $(\mathbb{Q}_p, +)$ and $(\mathbb{Q}_p,^\times \times)$.

For that end, let G be a locally compact topological abelian group. A *character* of G is a continuous group homomorphism $\chi : G \to \mathbb{C}^\times$. A character is called *unitary* if its image lie in the unit circle S^1 of $\mathbb{C}$. Notice that the torus S^1 is the maximal compact subgroup of $\mathbb{C}^\times$. Hence, by continuity, if G is a compact group then all its characters are unitary. We denote by $\widehat{G}$ the set of unitary character of the group G. This is called the Pontryagin dual of G (hence the dual notation). Clearly $\widehat{G}$ is itself a group when endowed with multiplication of characters. Moreover, we can define a natural topology on the group $\widehat{G}$ generated by the collection of sets

$$D_{K,U} := \{\chi \in \widehat{G}, \chi(K) \subset U\} \text{ for } K \text{ compact in } G \text{ and } U \text{ open in } S^1.$$

From the theory of Haar measures, it is known that every locally compact group has left and right Haar measures that are unique up to scalar multiplication. Let λ be a left Haar measure of the group G. Recall that this implies $\lambda(gA) = A$ for any measurable set in A (the σ-algebra here is the Borel σ-algebra of the topological group G). Let us start with an easy but interesting lemma.

LEMMA 3.1. — *If χ is a non-trivial character of a locally compact abelian group G. Then we have*

$$\int_G \chi(x)\lambda(dx) = 0$$

Proof. — If χ is non-trivial, there exists $g \in G$ such that $\chi(g) \neq 1$. Then, since λ is a left Haar measure, we get

$$\begin{aligned}\int_G \chi(x)\lambda(dx) &= \int_G \chi(gx)\lambda(gdx) \\ &= \int_G \chi(gx)\lambda(dx) \\ &= \chi(g)\int_G \chi(x)\lambda(dx).\end{aligned}$$

The result follows since $\chi(g) \neq 1$. □

This result is the generalization of the classical result

$$\int_0^{2\pi} e^{itx}dx = 0, \qquad \text{if } t \neq 0.$$

Now, we can define the Fourier transform of a function $f \in L^1(G)$ to be the function $\widehat{f}$ on the Pontryagin dual $\widehat{G}$ defined by

$$\widehat{f}(\chi) = \int_G f(g)\chi(g)\lambda(dg). \tag{3.1}$$

This definition depends obviously on the choice of the left Haar measure λ, but since these are unique up to scalar multiplication the Fourier transform is unique up to scalar multiplication.

The Pontryagin dual $\widehat{G}$ of a locally compact group G is also locally compact. Hence it also has a left Haar measure (again unique up to scalar multiplication). We can choose a left Haar measure $\widehat{\lambda}$ on $\widehat{G}$ such that for any $f \in L^1(G)$ such that $\widehat{f} \in L^1(\widehat{G})$ we have

$$f(g) = \int_{\widehat{G}} \widehat{f}(\chi)\chi(g^{-1})\widehat{\lambda}(d\chi).$$

This is the generalization of the Fourier inversion formula to locally compact groups. As is known for classical Fourier theory, when f is smooth enough (when f is in the Schwartz space for example), its Fourier transform is integrable and the Fourier inversion formula applies. We shall see the analogue of this property for $\mathbb{Q}_p$, in particular we will define the p-adic Schwartz space.

The Fourier transform can be extended to an isometry $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$. This is due to the so-called Plancherel theorem:

THEOREM 3.2. — *Let $\mathcal{F} : L^2(G) \cap L^1(G) \to L^2(\widehat{G}) \cap L^1(\widehat{G})$ defined by $F(f) = \widehat{f}$. Then $\mathcal{F}$ is an isometry, i.e,*

$$\int_G |f(g)|^2\lambda(dg) = \int_{\widehat{G}} |\widehat{f}(\chi)|\widehat{\lambda}(d\chi),$$

and can be extended into an isometry $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$.

Remark 3.1. — The Haar measures $\lambda, \widehat{\lambda}$ can be chosen so that the above isometry equation holds. This is the source of the usual scaling factor $\frac{1}{(2\pi)^d}$ in the classical Fourier theory.

Of course, no discussion of Fourier tranform is complete without mentioning convolution product. For $\phi, \varphi \in L^1(G)$ on can define the convolution product of φ and ϕ in the usual way

$$(\varphi * \phi)(g) = \int_G \varphi(h)\phi(h^{-1}g)\lambda(dh).$$

One has $\varphi * \phi \in L^1(G)$ and the usual formula

$$\widehat{\varphi * \phi}(\chi) = \widehat{\varphi}(\chi)\widehat{\phi}(\chi).$$

4. Characters of $(\mathbb{Q}_p, +)$

Since $(\mathbb{Q}_p, +)$ is a locally compact topological group, the theory of characters presented in the previous section applies. In this section we shall exhibit some interesting properties of p-adic Fourier analysis.

First, $\mathbb{Q}_p$ enjoys the nice property that all its characters are unitary. This facts, as we have seen, holds for compact groups in general but since $\mathbb{Q}_p$ is only locally compact we need to prove it.

Proposition 4.1. — *All the additive characters of* $\mathbb{Q}_p$ *are unitary.*

Proof. — If χ is a character of $\mathbb{Q}_p$ it is continuous. For any compact additive subgroup H of $\mathbb{Q}_p$ the restriction $\chi_{|H}$ is unitary. But since $\mathfrak{p}^n$ is a compact subgroup of $\mathbb{Q}_p$ for any $n \in \mathbb{Z}$ and $\mathbb{Q}_p = \cup_{n\in\mathbb{Z}}\mathfrak{p}^n$, we deduce that χ is also unitary. □

Let's start by defining the *fundamental character* χ of $(\mathbb{Q}_p, +)$. As we shall see, we can retrieve all the characters of $(\mathbb{Q}_p,)$ from this character (hence the terminology). We define $\chi : \mathbb{Q}_p \to S^1$ by

$$\chi(x) = \exp(2\pi i\omega(x)).$$

Lemma 4.1. — *The map* χ *is a character of* $(\mathbb{Q}_p, +)$.

Proof. — For $x, y \in \mathbb{Q}_p$ such that $v_p(x) \leqslant v_p(y)$ with

$$\omega(x) = \sum_{n=v_p(x)}^{-1} a_n \text{ and } \omega(y) = \sum_{n=v_p(y)}^{-1} b_n.$$

Obviously, if $x \in \mathbb{Z}_p$ and $y \in \mathbb{Z}_p$ the two previous sums are 0. We can then write

$$\omega(x+y) = \sum_{k=v_p(x)}^{-1} (a_k + b_k + \epsilon_{k-1})p^k$$

where the numbers ϵ_k are defined as follows:

$$\epsilon_{k-1} = \begin{cases} 1, & \text{if } a_{k-1} + b_{k-1} + \epsilon_{k-2} > p \\ 0, & \text{otherwise.} \end{cases}$$

where we take $a_k = 0$ whenever $k < v_p(x)$ and $b_k = 0$ whenever $k < v_p(y)$ and also $\epsilon_{k-1} = 0$ for $k = v_p(x)$. Comparing $\omega(x) + \omega(y)$ to $\omega(x+y)$, we notice that they are the same unless $a_{-1} + b_{-1} + \epsilon_{-2} > p$ in which case they differ by 1. So, in any case we have $e^{2\pi i \omega(x+y)} = e^{2\pi i(\omega(x)+\omega(y))}$ which means that $\chi(x+y) = \chi(x)\chi(y)$.

For continuity, simply notice that χ is trivial on $\mathbb{Z}_p$ which is a neighborhood of 0. Hence χ is continuous. Then it is indeed a character. □

The fact that χ is trivial on $\mathbb{Z}_p$ is a general fact for characters of $(\mathbb{Q}_p, +)$ in the following sense.

Proposition 4.2. — *If φ is a non-trivial character of $(\mathbb{Q}_p, +)$, then there exists an integer $n \in \mathbb{Z}$ such that φ is trivial on $\mathfrak{p}^n$ and non trivial on $\mathfrak{p}^{n-1}$.*

Proof. — Let B be the open ball of radius $1/2$ around 1 in $\mathbb{C}$. Obviously the only subgroup of $\mathbb{C}^\times$ contained in B is the trivial group $\{1\}$. By continuity of φ, there exists $n \in \mathbb{Z}$ such that $\varphi(\mathfrak{p}^n) \subset B$. Since φ is a group homomorphism and $\mathfrak{p}^n$ is a subgroup of $\mathbb{Q}_p$, its image $\varphi(\mathfrak{p}^n)$ is a subgroup of $\mathbb{C}^\times$ that lies inside B. Hence $\varphi(\mathfrak{p}^n) = \{1\}$, and the desired result follows by taking a minimal n. □

We can then define the *conductor* of a character φ to be the largest subgroup $\mathfrak{p}^n$ on which φ is trivial. For example, the conductor of the fundamental character χ is the group $\mathbb{Z}_p = \mathfrak{p}^0$.

Remark 4.3. — Let φ be a character with conductor $\mathfrak{p}^n$. Notice that, since for any $x \in \mathbb{Q}_p$ there exists $m \in \mathbb{Z}$ such that $p^m x \in \mathfrak{p}^n$, we have $\varphi(p^m x) = \varphi(x)^{p^m} = 1$. So the image of any character φ of $(\mathbb{Q}_p, +)$ lies in the subgroup of p^∞-units of $\mathbb{C}$, i.e. for any character $\varphi \in \widehat{\mathbb{Q}_p}$:

$$\varphi(\mathbb{Q}_p) \subset \mu_{p^\infty}(\mathbb{C}) := \{z \in \mathbb{C}, z^{p^n} = 1 \text{ for some } n \geqslant 0\}.$$

We now explain how to build other charcters from the fundamental character χ. This is simply by following the same recipe in classical Fourier theory on $(\mathbb{R}, +)$. We define the character χ_u for $u \in \mathbb{Q}_p$ as $\chi_u(x) = \chi(ux)$ for any $x \in \mathbb{Q}_p$. This analogue to the fact that all the characters $x \to e^{2i\pi tx}$ of $(\mathbb{R}, +)$ come from the character $x \to e^{2i\pi x}$.

For $u, v \in \mathbb{Q}_p$ with $\chi_u = \chi_v$ we have $\chi((u-v)x) = 1$ for all $x \in \mathbb{Q}_p$. Since the fundamental character χ is non-trivial we deduce that $u = v$. So the characters (χ_u) are distinct. Clearly the conductor of χ_u is simply $\mathfrak{p}^{-v_p(u)}$. Now we state the central theorem of this section.

THEOREM 4.2 (Tate's theorem). — *The Pontryagin dual of $\mathbb{Q}_p$ is the following*

$$\widehat{\mathbb{Q}_p} = \{\chi_u, u \in \mathbb{Q}_p\} \simeq \mathbb{Q}_p.$$

Sketch of the proof. — Let $A = \{\chi_u, u \in \mathbb{Q}_p\}$, this is obviously a multiplicative subgroup of $\widehat{\mathbb{Q}_p}$. We decompose the proof into four steps.

(1) The map $u \to \chi_u$ is a group isomorphism from $\mathbb{Q}_p$ to A. It's clearly surjective and injectivity follows from the previous discussion.
(2) The group A is dense in $\widehat{\mathbb{Q}_p}$. This is because if $x \in \mathbb{Q}_p$ is such that $\chi_u(x) = 1$ for all u then $x = 0$.
(3) The map $u \to \chi_u$ is bicontinuous. To see why, let $M > 0$ and $B = \{x \in \mathbb{Q}_p, |x|_p \leqslant M\}$ for a large M. For u close enough to 0 the the restriction $\chi_{|uB}$ is trivial so the character χ_u is close to the trivial character in the topology of $\widehat{\mathbb{Q}_p}$. On the other hand, if χ_u is close to the trivial character, then $\chi_{|uB}$ is trivial for a large M. Hence u is close to 0 in $\mathbb{Q}_p$.
(4) The group A is then a locally compact subgroup of $\widehat{\mathbb{Q}_p}$. This implies that A is complete. Since A is also dense in $\widehat{\mathbb{Q}_p}$, we deduce that $\widehat{\mathbb{Q}_p} = A$.

□

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5. Fourier analysis on $(\mathbb{Q}_p, +)$

Let us fix the Haar measure λ on $\mathbb{Q}_p$ such that $\lambda(\mathbb{Z}_p) = 1$ (notice that it is unique). This measure is translation invariant by definition and every compact set of $\mathbb{Q}_p$ has a finite measure with respect to λ. Also, $\lambda(\mathfrak{p}^n) = p^{-n}$ so any non-empty open set has positive measure.

PROPOSITION 5.1. — *We have the following:*

$$\int_{\mathfrak{p}^n} \chi(x)\lambda(dx) = \begin{cases} p^{-n} & \text{if } n \geqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. — This follows from the proof of Lemma 3.1 because the conductor of χ is $\mathbb{Z}_p = \mathfrak{p}^0$. □

To more analysis on $\mathbb{Q}_p$, we now introduce the p-adic Schwartz space.

DEFINITION 5.2. — *We define the space $C_c^\infty(\mathbb{Q}_p)$ of complex valued locally constant functions with compact support.*

This is called the Schwartz space of $\mathbb{Q}_p$. Obviously, any function $f \in C_c^\infty(\mathbb{Q}_p)$ is continuous and since f has compact support, then there exists $n \in \mathbb{Z}$ such that $\mathrm{supp}(f) \subset \mathfrak{p}^n$. Since $f \in C_c^\infty(\mathbb{Q}_p)$ is locally compact, then there also exists an integer n such that f is constant on cosets of $\mathfrak{p}^n$. This implies that f takes finitely many values. Finally, the space $C_c^\infty(\mathbb{Q}_p)$ is dense in each of the spaces $L^p(\mathbb{Q}_p)$ for $1 \leqslant p \leqslant \infty$. Now we can define the Fourier transform on $L^1(\mathbb{Q}_p)$.

DEFINITION 5.3. — *For $f \in L^1(\mathbb{Q}_p)$, we define the Fourier transform of f as the function on $\mathbb{Q}_p$ given by*

$$\widehat{f}(u) = \int_{\mathbb{Q}_p} f(x)\chi_u(x)\lambda(dx)$$

This is a particular case of the Definition 3.1. The difference is that in equation (3.1) the Fourier transform $\widehat{f}$ takes as arguments the characters in $\widehat{\mathbb{Q}_p}$. But since, thanks to theorem Tate's Theorem 4.2, we have $\widehat{\mathbb{Q}_p} \simeq \mathbb{Q}_p$ via $u \mapsto \chi_u$ we can consider that $\widehat{f}$ is a function on $\mathbb{Q}_p$ instead of $\widehat{\mathbb{Q}_p}$.

THEOREM 5.1. — *The map $f \mapsto \widehat{f}$ is a bijection from the Schwartz space $C_c^\infty(\mathbb{Q}_p)$ onto itself. Moreover, a function $f \in C_c^\infty(\mathbb{Q}_p)$ is supported on $\mathfrak{p}^m$ and constant on cosets of $\mathfrak{p}^n$ with $n \geqslant m$ if and only if its transform $\widehat{f}$ is supported on p^{-n} and constant on the cosets of $\mathfrak{p}^{-m}$.*

Proof. — Let $f \in C_c^\infty(\mathbb{Q}_p)$ with support in $\mathfrak{p}^m$ and $n \geqslant m$ such that f is constant on the cosets of $\mathfrak{p}^n$. Obviously, If $m = n$, the function f is constant. Now let $v \in \mathfrak{p}^{-m}$, then we have

$$\begin{aligned}\widehat{f}(u+v) &= \int_{\mathfrak{p}^m} f(x)\chi_{u+v}(x)\lambda(dx)\\ &= \int_{\mathfrak{p}^m} f(x)\chi(ux+vx)\lambda(dx)\\ &= \int_{\mathfrak{p}^m} f(x)\chi(ux)\chi(vx)\lambda(dx)\end{aligned}$$

But since for $x \in \mathfrak{p}^m$, we have $vx \in \mathbb{Z}_p$, and χ is trivial on $\mathbb{Z}_p$ we deduce that

$$\widehat{f}(u+v) = \int_{\mathfrak{p}^m} f(x)\chi(ux)\lambda(dx) = \widehat{f}(u).$$

Hence, f is constant on cosets of $\mathfrak{p}^{-m}$. Now, for $y \in \mathfrak{p}^n$, since f is constant on cosets of $\mathfrak{p}^n$, we have

$$\begin{aligned}\widehat{f}(u) &= \int_{\mathbb{Q}_p} f(x)\chi(ux)\lambda(dx) \\ &= \int_{\mathbb{Q}_p} f(x)\chi(u(x-y))\lambda(dx) \\ &= \chi(-uy)\int_{\mathbb{Q}_p} f(x)\chi(ux))\lambda(dx) = \chi(-uy)\widehat{f}(u).\end{aligned}$$

So if $u \notin \mathfrak{p}^{-n}$, this means that $uy \notin \mathbb{Z}_p$ and hence $\chi(-uy) \neq 1$. This implies that $f(u) = 0$ whenever $u \notin \mathfrak{p}^{-n}$. Conversely, we proceed in the same way. □

Let us not compute some examples of Fourier tranform. Let $\mathbf{1}_n$ the indicator of the set $\mathfrak{p}^n$ for $n \in \mathbb{Z}$. Then, for $u \in \mathbb{Q}_p$ we have thanks to Lemma 5.1

$$\widehat{\mathbf{1}_n}(u) = \int_{\mathfrak{p}^n} \chi(ux)\lambda(dx) = \frac{1}{|u|_p}\int_{\mathfrak{p}^{n+v_p(u)}} \chi(x)\lambda(dx) = \frac{p^{-n-v_p(u)}}{|u|_p}\mathbf{1}_{-n}(u).$$

Hence we deduce that that

$$\widehat{\mathbf{1}_n} = p^{-n}\mathbf{1}_{-n}.$$

Now, let $a \in \mathbb{Q}_p$ and $n \in \mathbb{Z}$. Let $\mathbf{1}_{a,n}$ be the indicator function of the coset $a + \mathfrak{p}^n$. Since any function f in the Schwartz space $C_c^\infty(\mathbb{Q}_p)$ is a linear combination of the functions $\mathbf{1}_{a,n}$, it suffices to compute the Fourier transforms of these elementary functions to get the transform of f.

Proposition 5.4. — *For $u \in \mathbb{Q}_p$, we have :*

$$\widehat{\mathbf{1}_{a,n}}(u) = p^{-n}\chi_a(u)\mathbf{1}_{-n}(u).$$

Proof. — This follows easily from the change of variable $u \mapsto a + u$ and from the fact that $\widehat{\mathbf{1}_n} = p^{-n}\mathbf{1}_{-n}$. □

So, if $f \in C_c^\infty(\mathbb{Q}_p)$ such that f is constant on cosets of $\mathfrak{p}^n$ we can write $f = \sum_{k=1}^{\ell} c_k \mathbf{1}_{a_k,n}$. Then we get the following

$$\widehat{f}(u) = \sum_{k=0}^{\ell} c_k \widehat{\mathbf{1}_{a_k,n}}(u) = p^{-n}\left(\sum_{k=0}^{\ell} c_k \chi(a_k u)\right)\mathbf{1}_{-n}(u).$$

Bibliography

[AZ01] Sergio Albeverio and Xuelei Zhao. A decomposition theorem for Lévy processes on local fields. *J. Theoret. Probab.*, 14(1):1–19, 2001.

[Cas86] John William Scott Cassels. *Local fields*, volume 3. Cambridge University Press Cambridge, 1986.

[CR66] Charles W Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras*, volume 356. American Mathematical Soc., 1966.

[EL07] Steven N Evans and Tye Lidman. Expectation, conditional expectation and martingales in local fields. *Electronic Journal of Probability*, 12(17):498–515, 2007.

[EP05] Antonio J Engler and Alexander Prestel. *Valued fields*. Springer Science & Business Media, 2005.

[Eva01] Steven N Evans. Local fields, gaussian measures, and brownian motions. *Topics in probability and Lie groups: boundary theory*, 28:11–50, 2001.

[GK12] Robert Todd Gregory and Edayathumangalam V Krishnamurthy. *Methods and applications of error-free computation*. Springer Science & Business Media, 2012.

[Khr90] A Yu Khrennikov. Mathematical methods of non-archimedean physics. *Russian Mathematical Surveys*, 45(4):87, 1990.

[Khr13] Andrei Y Khrennikov. *p-Adic valued distributions in mathematical physics*, volume 309. Springer Science & Business Media, 2013.

[Sch84] WH Schikhof. *Ultrametric Calculus (Cambridge Studies in Advanced Mathematics, 4)*. Cambridge University Press, Cambridge, 1984.

[Sch07] Wilhelmus Hendricus Schikhof. *Ultrametric Calculus: an introduction to p-adic analysis*, volume 4. Cambridge University Press, 2007.

[Ser13] Jean-Pierre Serre. *Local fields*, volume 67. Springer Science & Business Media, 2013.

[vR78] Arnoud CM van Rooij. *Non-Archimedean functional analysis*. Dekker New York, 1978.

[VVZ94] Vasilii Sergeevich Vladimirov, Igor Vasilievich Volovich, and Evgenii Igorevich Zelenov. *p-adic Analysis and Mathematical Physics*. World Scientific, 1994.

[Wei13] André Weil. *Basic number theory.*, volume 144. Springer Science & Business Media, 2013.