BESSEL PROCESSES AND RAY-KNIGHT THEOREMS

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1. Introduction and notations

In these notes we discuss general Bessel processes and their properties. We denote by $\mathbf{W}$ the space $C(\mathbb{R}^+, \mathbb{R})$ of continuous functions on the positive half line and by X the coordinate process on this space (in Section 2 we will define a family of probability measures on this space). For an integer d we denote by $B^{(d)}$ "the" d-dimensional Brownian motion and $r_t^{(d)} = \left\|B_t^{(d)}\right\|_2$ the euclidean norm process of $B^{(d)}$ known as the Bessel process of order d. It is known that the process $r_t^{(d)}$ satisfies the following SDE

$$dr_t{}^{(d)} = \frac{d-1}{2r_t^{(d)}}dt + dB'_t$$

where B' is a certain linear Brownian motion. We write r_t instead of $r_t^{(d)}$ when there is no ambiguity on the dimension. This defines a countable family of probability measures on the space $\mathbf{W}$ that we will generalize a little further in our discussion. Before we do so we need to recall some results and concepts that we will encounter throughout these notes.

***Theorem 1.1* (Uniqueness of strong solution of SDEs).** — *Consider the SDE*

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t \quad (*)$$

such that $|b(t,x) - b(t,y)| \leq K|x-y|$ *for any* $x, y \in \mathbb{R}$ *and* $t \geq 0$ *and* $|\sigma(t,x) - \sigma(t,y)| \leq h(|x-y|)$ *where the function* h *satisfies the conditions*

$$h \text{ is stricrly increasing , } h(0) = 0 \text{ and } \int_0^\epsilon h^{-2}(u)du = +\infty \text{ for all } \epsilon > 0$$

Then there exists at most one strong solution for the equation $(*)$

Theorem 1.2 **(Yamada Watanabe).** — *The existence of weak solutions for a stochastic differential equation for which strong uniqueness holds implies existence and uniqueness of the strong solution.*

2. Bessel processes

In order to extend the definition of Bessel processes we first notice that in dimension d one has the following relation with r_t and $B^{(d)}$ using Itô's formula:

$$r_t^2 = r_0^2 + 2\sum_{i=1}^{d}\int_0^t B_s^i dB_s^i + d.t$$

We know that r_t is positive for all $t > 0$ almost surely when $d > 1$ and that for $d = 1$ the set of zeros of r_t has 0 Lebesgue measure so that we can make sense of the process

$$\beta_t = \sum_{i=1}^{d}\int_0^t \frac{B_s^i}{r_s} dB_s^i$$

Using the Itô's isometry we can get $\langle \beta, \beta \rangle_t = t$ then by Lévy's theorem β is a linear Brownian motion. Finally notice that we have again by Itô 's formula

$$r_t^2 = r_0^2 + 2\int_0^t r_s d\beta_s + d.t$$

We regard Bessel processes as solutions of the previous SDEs indexed by d. To extend this definition, for real numbers $\delta \geq 0$ and $x \geq 0$ consider the SDE

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|} d\beta_s + \delta t$$

A natural question to ask is whether this equation has solutions or not. The answer is yes, there exists a unique strong solution for this SDE: since $|\sqrt{x} - \sqrt{y}| \leq \sqrt{|x-y|}$ one can invoke a result of Yamada and Watanabe on the existence of strong solutions for SDEs (see section 3 of chapter IX in [RY13]). The solution is $Z \equiv 0$ when $x = \delta = 0$ and by the comparison theorems (see chapter IX in [RY13]) we then get $Z_t \geq 0$ a.s for all cases of δ and x. Then one can get rid of the absolute value in the above equation and state the

Definition 2.1. — For every $\delta \geq 0$ and $x \geq 0$ the unique strong solution of the equation

$$Z_t = x + 2\int_0^t \sqrt{|Z_s|} d\beta_s + \delta t$$

is called the square of δ-dimensional Bessel process started at x and we denote it by $BESQ_\delta(x)$. The real number δ is called the dimension and we call $\nu := \frac{\delta}{2} - 1$ the *index* of this process. We may write $BESQ^\nu$ when we want to index by ν instead of δ.

This defines a two parameter family of probability measures on $\mathbf{W}$ which we denote by $\mathbb{Q}_x^\delta$ (we write $\mathbb{Q}_x^{(\nu)}$ to use the index instead of the dimension) which coincides with the squared modulus of the Brownian motion when δ is an integer. We give a first result concerning this family of distributions which is trivial when d is an integer.

Proposition 2.2. — *For $\delta, \delta' \geq 0$ and $x, x' \geq 0$ one has*

$$\mathbb{Q}_x^\delta * \mathbb{Q}_{x'}^{\delta'} = \mathbb{Q}_{x+x'}^{\delta+\delta'}$$

By $*$ here we mean the convolution product, more precisely $\mathbb{Q}_x^\delta * \mathbb{Q}_{x'}^{\delta'}$ is the push-forward of the measure $\mathbb{Q}_x^\delta \otimes \mathbb{Q}_{x'}^{\delta'}$ on the space $\mathbf{W}^2$ by the map $\mathbf{W}^2 \to \mathbf{W}, (w, w') \mapsto w + w'$.

Proof. Let β, β' be two independent linear Brownian motions and Z, Z' the corresponding solutions for (x, δ) and (x', δ') and $X = Z + Z'$. Then obviously

$$X_t = x + x' + 2\int_0^t \left(\sqrt{Z_s} d\beta_s + \sqrt{Z_s'} d\beta_s'\right) + (\delta + \delta')t$$

Now let β'' be a third Brownian motion independent of the first two. Define γ as

$$\gamma = \int_0^t 1_{X_s > 0} \frac{\sqrt{Z_s} d\beta_s + \sqrt{Z_s'} d\beta_s'}{\sqrt{X_s}} + \int_0^t 1_{X_s = 0}\, d\beta_s''$$

We can show that $\langle \gamma, \gamma \rangle_t = t$ which means again by Lévy's theorem that γ is a linear Brownian motion. Finally we have

$$X_t = (x + x') + 2\int_0^2 \sqrt{X_s} d\gamma_s + (\delta + \delta')t$$

This finishes the proof since it implies that X has distribution $\mathbb{Q}_{x+x'}^{\delta+\delta'}$. □

Remark 2.3. — Notice that this result shows that the $\mathbb{Q}_x^\delta$'s are infinitely divisible laws on $\mathbf{W}$. These are not the only distributions that satisfy this kind of identity (one can take a look at exercise 1.13 448 in [RY13]).

This result, intuitive as it is, proves to be useful as the following corollary and discussion show.

Corollary 2.4. — *If μ is a measure on $\mathbb{R}^+$ with $\int_{\mathbb{R}^+}(1+t)d\mu(t)$, then there exists two positive real numbers A_μ, B_μ with*

$$\mathbb{E}_{\mathbb{Q}_x^\delta}\left[\exp\left(-\int_0^\infty X_t d\mu(t)\right)\right] = A_\mu^x B_\mu^\delta$$

where X is the coordinate process.

Proof. We define $\phi(x,\delta) = \mathbb{E}_{\mathbb{Q}_x^\delta}\left[\exp\left(-\int_0^\infty X_t d\mu(t)\right)\right]$ and establish a functional equation on ϕ. First notice that by using Jensen's inequality one can see that $\phi(x,\delta) > 0$. We also have thanks to the previous result that

$$\phi(x+x', \delta+\delta') = \phi(x,\delta)\phi(x',\delta')$$

This yields the equation $\phi(x,\delta) = \phi(x,0)\phi(0,\delta)$ and thus the separation of x and δ. The functions $\phi(0,.), \phi(.,0)$ are multiplicative and monotone hence they are exponential. This finishes the proof. □

Now we give a more concrete application of this result by choosing $\mu = \lambda\epsilon_t$ where ϵ_t is the Dirac measure in t. We get

$$\mathbb{E}_{\mathbb{Q}_{x,1}}[\exp(-\lambda X_t)] = \mathbb{E}_{\sqrt{x}}[e^{-\lambda B_t^2}]$$

An easy integration show that $\mathbb{E}_{\sqrt{x}}[e^{-\lambda B_t^2}] = (1+2\lambda)^{-1/2}\exp(-\lambda x/(1+2\lambda t))$ and hence we deduce

$$\mathbb{E}_{\mathbb{Q}_{x,\delta}}[\exp(-\lambda X_t)] = (1+2\lambda)^{-\delta/2}\exp(-\lambda x/(1+2\lambda t))$$

Inverting this Laplace transform gives us the semi-group of $BESQ_\delta$

Corollary 2.5. — *For $\delta > 0$, the semigroup of $BESQ_\delta$ has a density in y given by*

$$q_t(x,y) = \frac{1}{2}\left(\frac{y}{x}\right)^{\nu/2}\exp\left(\frac{-(x+y)}{2t}\right)J_\nu(\sqrt{xy}/t)$$

where J_ν is the Bessel function of index ν

When $x = 0$ one has

$$a_t(0,y) = (2t)^{-\nu-1}\,\Gamma(\delta/2)\exp\left(\frac{-y}{2t}\right)y^\nu$$

As a consequence the process $BESQ_\delta$ is a Feller process. Notice that for a continuous function f on $\mathbb{R}^+$ the map $\mathbb{E}_{\mathbb{Q}_{x,\delta}}[f(X_t)]$ is continuous in both x and t (Stone Weierstrass + special case $f(x) = e^{-\lambda x}$) so one may apply the results on that we have already seen previously (see chapter III in [RY13]) so conclude that this is indeed a Feller process. Here are a few observations on the behavior of these processes.

From the comparison theorem for SDEs and the facts we have already established for Brownian motion we get

i. For $\delta \geq 3$ the process $BESQ^\delta$ is transient and for $\delta \leq 2$ it is recurrent.
ii. For $\delta \geq 2$ the set $\{0\}$ is polar and for $\delta \leq 1$, it is reached a.s. Furthermore for $\delta = 0$ the origin is an absorbing point.

It remains to say something about the case of small δ. But if one considers

$$s_\nu(x) = \begin{cases} -x^{-\nu} \text{ if } \nu > 0 \\ x^{-\nu} \text{ if } \nu < 0 \end{cases} \quad \text{and } s_0(x) = \log(x)$$

and T the hitting time of 0 then Itô's formula shows that $s_\nu(X)^T$ is a local martingale under $\mathbb{Q}_x^\delta$. The point 0 is then reached almost surely for $0 \leq \delta < 2$ (check exercise III.3.21 [RY13]).

Proposition 2.6. — *For $\delta = 0$, the point 0 is absorbing. For $< \delta < 2$, the point 0 is instantaneously reflecting.*

The first point is trivial since when $x = \delta = 0$ the solution is the zero process.

Proof. For $0 < \delta < 2$ if X is a $BESQ_\delta$ then it is a semi-martingale (just by definition from the SDE that X satisfies). We have the local time

$$L_t^0(X) = 2\delta \int_0^t 1_{X_s=0} ds$$

and $\langle X, X\rangle_t = 4X_t dt$ so the occupation formula that we have seen in the local times chapter gives

$$\int_0^\infty (4a)^{-1} L_t^a(X) da = \int_0^t 1_{0<X_s} (4X_s)^{-1} d\langle X, X\rangle_s = \int_0^t 1_{0<X_s} ds \leq t$$

Then we deduce that $L_t^0(X) = 0$ for all t. Hence X spends not times at 0. □

We recall the scaling properties of Brownian motion. If $B_t^x = x + B_t$ then for any $x > c$ the processes $B_{c^2t}^x$ and $cB_t^{x/c}$ have the same distribution. The same kind of scaling applies to $BESQ_\delta$.

Proposition 2.7. — *If X is a $BESQ_\delta(x)$ then for any $c > 0$, the process $c^{-1}X_{ct}$ is a $BESQ_\delta(x/c)$.*

Proof. A change of variable in the SDE defining $BESQ_\delta(x)$ gives

$$c^{-1}X_{ct} = c^{-1}x + 2\int_0^t (c^{-1}X_c s)^{1/2} c^{-1/2} dB_{cs} + \delta t$$

which finishes the proof. □

We go back to corollary 2.4 to explain how one can compute the constants A_μ, B_μ which will help us compute the transform of some Brownian functionals. We recall from the Local times chapter in [RY13] that for a Radon measure μ the equation $\phi'' = \phi\mu$ (is the sense of distribution) has a unique solution which is unique solution ϕ_μ which is positive and non increasing on $\mathbb{R}^+$ and we have $\phi_\mu(0) = 1$. Furthermore ϕ_μ is convex, so it's right-derivative ϕ'_μ exists and is in $[0, 1]$ (even better $\phi_\mu(\infty) < 1$ otherwise $\mu = 0$ and $\phi_\mu = 0$). From here on in suppose that $\int_{\mathbb{R}^+}(1 + x)d\mu(x) < +\infty$ (as we will see this implies $\phi_\mu(\infty) > 0$). Let

$$X_\mu = \int_0^\infty X_t d\mu(t)$$

Theorem 2.8. — *In the previous setup we have*

$$\mathbb{E}_{\mathbb{Q}_{x,\delta}}\left[\exp\left(-\frac{1}{2}X_\mu\right)\right] = \phi_\mu(\infty)^{\delta/2}\exp\left(\frac{x}{2}\phi'_\mu(0)\right)$$

Proof. ϕ'_μ is a right continuous and increasing then $F_\mu = \frac{\phi'_\mu}{\phi_\mu}$ is right continuous and of finite variation . Then using integration by parts (or Itô 's formula for the product function) we get

$$F_\mu(t)X_t = F_\mu(0)x + \int_0^t F_\mu(s)dX_s + \int_0^t X_s dF_\mu(s)$$

On the other hand one has

$$\begin{aligned}\int_0^t X_s dF_\mu(s) &= \int_0^t X_s \frac{d\phi'_\mu(s)}{\phi_\mu(s)} - \int_0^t X_s \frac{\phi'_\mu(s)}{\phi^2_\mu(s)} d\phi_\mu(s) \\ &= \int_0^t X_s d\mu(s) - \int_0^t X_s F^2_\mu(s) ds\end{aligned}$$

Hence, since $M_t = X_t - \delta t$ is a $\mathbb{Q}_{x,\delta}$ continuous local martingale, the process

$$\mathcal{E}_t = \exp\left(\frac{1}{2}\int_0^t F_\mu(s)dM_s - \frac{1}{2}\int_0^t F^2_\mu(s)ds\right)$$

is a continuous local martingale and we have

$$\mathcal{E}_t = \exp\left(\frac{1}{2}\left[F_\mu(t)X_t - F_\mu(0)x - \delta\log(\phi_\mu(t))\right] - \frac{1}{2}\int_0^t X_s d\mu(s)\right)$$

This local martingale is bounded on $[0,T]$ for any $T > 0$ because F is negative and X is positive and we get

$$\mathbb{E}[Z^\mu_t] = \mathbb{E}[Z^\mu_0] = 1$$

as $t \to \infty$ we get the desired result as the following argument explains:

Proposition 2.7 implies that $\frac{X_t}{t}$ converges in distribution as $t \to \infty$ and

$$\phi'_\mu(x) = -(\phi_\mu\mu)((x,+\infty)) \text{ and } 0 < aF_\mu(a) \leq \int_a^\infty x d\mu(x) \xrightarrow[a\to\infty]{} 0$$

This implies that $F_\mu(t)X_t$ converges in probability to 0 and this finishes the proof. □

This result gives an easy proof of the Cameron-Martin Formula which is

$$\mathbb{E}\left[\exp\left(-\lambda\int_0^1 B_s^2 ds\right)\right] = \frac{1}{\sqrt{\cosh(\sqrt{2\lambda})}}$$

This can be obtained by picking $x = 0$ and $\delta = 1$ in the following equation

Corollary 2.9. —

$$\mathbb{E}_{\mathbb{Q}_{x,\delta}}\left[\exp\left(-\frac{b^2}{2}\int_0^1 X_s ds\right)\right] = \cosh(b)^{-\delta/2} exp\left(-\frac{1}{2}xb\tanh(b)\right)$$

Proof. We need to compute the solution ϕ_μ of the equation $\phi'' = \phi\mu$ when $\mu(ds) = b^2 ds$ on $[0, 1]$. It's not too hard to show that $\phi_\mu(t) = \alpha\cosh(bt) + \beta\sinh(bt)$ and the initial condition gives $\alpha = \phi_\mu(0)1$. Since ϕ is constant on $[1, +\infty[$ and ϕ'_μ is continuous we must have $\phi'_\mu(1) = 0$ which means that

$$b\sinh(b) + \beta b\cosh(b) = 0$$

this gives $\beta = -\tanh(b)$. Then we deduce that $\phi_\mu(t) = \cosh(bt) - \tanh(b)\sinh(bt)$ on $[0, 1]$. Hence $\phi(\infty) = \phi_\mu(1) = \cosh(b)^{-1}$ and $\phi'_\mu(0) = -b\tanh(b)$ □

So far we have only discussed the square of Bessel processes. We now discuss Bessel processes themselves which just amounts to applying the homeomorphism of $\mathbb{R}^+$ given by $x \mapsto \sqrt{x}$. This means that since X is a Markov process under $\mathbb{Q}_{x,\delta}$ then the process $\sqrt{X}$ also is.

Definition 2.10. — The square root of $BESQ_\delta(x^2)$ is what's called the Bessel process if dimension δ started at x and we denote this process by $BES_\delta(x)$. We denote it's distribution on $\mathbf{W}$ by $\mathbb{P}_x^\delta$.

Most of the results of the previous discussions can be stated for the process $BES_\delta(x)$. Namely the results on transience and recurrence and one can also compute the semi-group of this process just from that of $BESQ_\delta(x^2)$. This shows that $BES_\delta(x)$ is also a Feller process. As we have seen in the introduction for integer values of δ the process $BES_\delta(x)$ satisfies the SDE

$$dX_t = x + d\beta_t + \frac{\delta - 1}{2}\frac{1}{X_t}dt$$

Here is a scaling result for this process

Proposition 2.11. — *BES_δ has the same scaling property as Brownian motion.*

Proof. Not very hard to show from the SDE. □

References

[RY13] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293. Springer Science & Business Media, 2013.

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