$X_i \overset{iid}{\sim} P$, what will be dist of $X_{(i)}$??

assume X_i is continuous r.v. have density f w.r.t lebesgue measure

$\Rightarrow$ we can ignore $X_i = X_j$ case.

Thm 1.

$X_1, \cdots, X_n \longrightarrow X_{(1)} < \cdots < X_{(n)}$

$$Y = (X_{(1)}, \cdots, X_{(n)})^t \Rightarrow \mathrm{pdf}_Y(y_1, \cdots, y_n) = n!\, f(y_1) \cdots f(y_n)\, I_{(y_1 < \cdots < y_n)}$$

(Pf)

$$\mathcal{X} = \{(x_1, \cdots, x_n)^t \mid f(x_i) > 0\} \to \mathcal{Y} = \{(y_1, \cdots, y_n)^t \mid f(y_i) > 0,\ y_1 < \cdots < y_n\}$$

u

$\downarrow$

$n! - 1$ correspondence

let $u^\pi(x_1, \cdots, x_n) = (x_{\pi(1)}, \cdots, x_{\pi(n)})$ for $\pi \in S_n$

$$\Rightarrow \left| \frac{\partial u^\pi_j}{\partial x_i} \right| = |e_{\pi(1)}, \cdots, e_{\pi(n)}| = 1$$

$\Rightarrow$ by change of variable formula

Thm 2

① $$\mathrm{pdf}_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x) (1 - F(x))^{n-r}$$

② $$\mathrm{pdf}_{X_{(r)}, X_{(s)}}(x, y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} F(x)^{r-1} f(x) (F(y) - F(x))^{s-1-r} f(y) (1 - F(y))^{n-s}$$

(Pf) ✓ $$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(y_1) \cdots f(y_{r-1})\, I(y_1 < \cdots < y_{r-1} < x)\, dy_1 \cdots dy_{r-1}$$

$$= \frac{1}{(r-1)!} \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} f(y_1) \cdots f(y_{r-1})\, dy_1 \cdots dy_{r-1} = \frac{1}{(r-1)!} F(x)^{r-1}$$

✓ $$\int \cdots \int f(y_{s+1}) \cdots f(y_n)\, I(x < y_{s+1} < \cdots < y_n)\, dy_{s+1} \cdots dy_n$$

$$= \frac{1}{(n-s)!} \int_{x}^{\infty} \cdots \int_{x}^{\infty} f(y_{s+1}) \cdots f(y_n)\, dy_{s+1} \cdots dx_n = \frac{1}{(n-s)!} (1 - F(x))^{n-r}$$

Lemma1 $\quad X_i \overset{iid}{\sim} Exp(1)$

$$Z_1 = nX_{(1)}, \quad Z_2 = (n-1)(X_{(2)} - X_{(1)}), \cdots, Z_n = X_{(n)} - X_{(n-1)}$$

$\Rightarrow Z_1, \cdots, Z_n \overset{iid}{\sim} Exp(1)$

(pf) $\quad Y = (X_{(1)}, \cdots, X_{(n)}) \qquad f_Y(y_1, \cdots, y_n) = n! e^{-(y_1 + \cdots + y_n)} I_{(0 < \cdots < y_n)}$

$u(y) = (ny_1, \cdots, y_n - y_{n-1})$

$u^{-1}(z) = \left(\frac{1}{n}z_1, \frac{1}{n}z_1 + \frac{1}{n-1}z_2, \cdots, \frac{1}{n}z_1 + \cdots + z_n\right)$

$|J_{u^{-1}}| = \frac{1}{n!}$

$$\Rightarrow f_Z(z_1, \cdots, z_n) = \frac{1}{n!} \cdot n! e^{-\left(\frac{1}{n}z_1 + \left(\frac{1}{n}z_1 + \frac{1}{n-1}z_2\right) + \cdots + \left(\frac{1}{n}z_1 + \cdots + z_n\right)\right)}$$

$$= e^{-(z_1 + \cdots + z_n)} I_{(z_i > 0)}$$

$\therefore Z_i \overset{iid}{\sim} exp(1)$

Lemma 2.

X has cdf F. $\quad$ (a) $F(X) \sim U(0,1)$

(b) $F^{-1}(U) \overset{d}{\equiv} X$ when $U \sim unif[0,1]$

Lemma3

$U_{(1)} < \cdots < U_{(n)}$ from $U(0,1)$

$Y_{(1)} < \cdots < Y_{(n)}$ from $exp(1)$

$$\Rightarrow (1 - e^{-y}) I_{(y>0)} = G(y) \quad \text{then} \quad G(Y_{(r)}) \overset{d}{\equiv} U_{(r)}$$

Thm3 X has cdf F, $h(y) = F^{-1}(1-e^{-y}) I_{(0,\infty)}(y)$

$$\Rightarrow X_{(r)} \overset{d}{\equiv} h\left(\frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1} Z_r\right) \quad \text{where } Z_r \overset{iid}{\sim} \exp(1)$$

(pf) $X_{(r)} \equiv F^{-1}(U_{(r)})$, $U_{(r)} \equiv 1-e^{-Y_{(r)}}$, $Y_{(r)} \equiv \frac{1}{n}Z_1 + \cdots + \frac{1}{n-r+1}Z_r$

Lemma 4. $U_{(r)} \sim \text{Beta}(r, n-r+1)$

Thm 4. $X_{(r)} \equiv F^{-1}(Z)$ when $Z \sim \text{Beta}(r, n-r+1)$

Cor. $(U_{(r)}/U_{(r+1)})^r \sim U(0,1)$ when $U_{(n+1)} = 1$

① $U_{(r)} \overset{d}{\equiv} 1-U_{(n+1-r)} \Rightarrow -\log U_{(r)} \overset{d}{\equiv} -\log(1-U_{(n+1-r)})$

② $-\log U_{(r)} \overset{d}{\equiv} -\log(1-(1-e^{-Y_{(n-r+1)}}))$

$\overset{d}{\equiv} Y_{(r)} \overset{d}{\equiv} \frac{1}{n}Z_1 + \cdots + \frac{1}{r}Z_{n-r+1}$

③ $-\log(U_{(r)}/U_{(r+1)})^r \overset{d}{\equiv} (Z_{n-r+1}) \sim \text{Exp}(1)$

④ $\therefore (U_{(r)}/U_{(r+1)})^r \sim U(0,1)$

now fix $\alpha \in (0,1)$. let $r_n = \lfloor \alpha n \rfloor$

$\Rightarrow$ Claim $\sqrt{n}\,(X_{(r_n)} - F^{-1}(\alpha)) \xrightarrow{d} N(0, \alpha(1-\alpha) f(F^{-1}(\alpha))^2)$

(pf). let $Y_n = \frac{1}{n} Z_1 + \cdots + \frac{1}{n-r_n+1} Z_{r_n}$

let $W_n = \sqrt{n}\,(Y_n - (-\log(1-\alpha))) / \sqrt{\alpha/(1-\alpha)}$

$$\text{cgf}\, Y_n(s) = \sum_{k=1}^{r_n} \text{cgf}\, Z_k\left(\frac{1}{n-k+1}, s\right)$$

$$= \sum_{k=1}^{r_n} \left(-\log\left(1 - \frac{s}{n-k+1}\right)\right)$$

$$\simeq \sum \left(\frac{s}{n-k+1}\right) + \frac{1}{2}\left(\frac{s}{n-k+1}\right)^2$$

$$\simeq s \int_0^\alpha \frac{1}{1-x}\,dx + s^2 \cdot \frac{1}{2n} \int_0^\alpha \frac{1}{(1-x)^2}\,dx$$

$$= -s\log(1-\alpha) + \frac{s^2}{2n}\frac{\alpha}{1-\alpha}$$

$\Rightarrow$ $\text{cgf}_{W_n}(t) = \frac{1}{2}t^2 + o(1)$

$\therefore$ $\sqrt{n}\,(Y_n - (-\log(1-\alpha))) \xrightarrow{d} N(0, \alpha/(1-\alpha))$

by delta method

$$\sqrt{n}\,(\underbrace{h(Y_n)}_{X_{(r_n)}} - \underbrace{h(-\log(1-\alpha))}_{F^{-1}(\alpha)}) \xrightarrow{d} N(0, \underbrace{\dot{h}(-\log(1-\alpha))^2 \cdot \frac{\alpha}{1-\alpha}}_{\frac{\alpha(1-\alpha)}{f(F^{-1}(\alpha))^2}})$$