- moment generating ftn. Cumulant generating function
- Basic dist / properties
- Change of variable formula : Beta, Chi-sq dist
- Quadratic form of normal dist

Def (moment generating function) given r.v X,

if $\exists h>0$ s.t $M(t) := \mathbb{E}(e^{tX}) < \infty$ for all $t \in (-h, h)$

we say M is the moment generating function of r.v. X

Thm (a) if r.v X has m.g.f, then $\mathbb{E}X^k$ exists for all k

and $\mathbb{E}(X^k) = M^{(k)}(0)$, and $M(t) = \sum_{k=0}^{\infty} \frac{\mathbb{E}(X^k) t^k}{k!}$ for $t \in (-\frac{h}{3}, \frac{h}{3})$

(b) if r.v. X, Y has mgf and if $\exists h$ s.t $M_X(t) = M_Y(t)$ f $t \in (-h, h)$

then X, Y has same distribution

(pf) (b) is beyond the scope of th lecture

fix $t \in (-h, h)$

(a) note that $e^a = \sum_{k=0}^{\infty} a^k / k!$ $\Rightarrow$ $|tx|^k / k! \le e^{|tx|} \le e^{tx} + e^{-tx}$

$\therefore$ $|t|^k \, \mathbb{E}|X|^k / k! \le M(t) + M(-t) < \infty$

$\therefore$ $\mathbb{E}X^k$ exists

also, by taylor thm

$$\left| e^{tx} - \sum_{k=0}^{n-1} \frac{(tx)^k}{k!} \right|$$

$$= \left| \frac{1}{(n-1)!} \int_0^1 (1-u)^{n-1} e^{utx} \, du \cdot (tx)^n \right|$$

$$\le \int_0^1 (1-u)^{n-1} \underbrace{e^{utx} \frac{|tx|^{n-1}}{(n-1)!} \frac{|tx|}{1!}}_{\le e^{|tx|}} \, du \le \frac{1}{n} \left(e^{3tx} + e^{-3tx}\right)$$

$$\therefore \left| M(t) - \sum_{k=0}^{n-1} \mathbb{E}X^k \cdot t^k / k! \right| \le \frac{1}{n} \left(M(3t) + M(-3t) \right)$$

$n \to \infty$ finishes the proof

* suppose $M(t)$ exists in $(-h, h)$

define $C(t) := \log M(t) = \log \mathbb{E}(e^{tX})$

note : $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$

$M(t) = 1 + m_1 t + \frac{m_2}{2} t^2 + \frac{m_3}{6} t^3 + \cdots$

$\Rightarrow C(t) = (m_1 t + \frac{m_2}{2} t^2 + \cdots) - (\quad " \quad)^2/2 + (\quad " \quad)^3/3$

$= m_1 t + \frac{1}{2}(m_2 - m_1^2) t^2 + \cdots$

$C_r(X) := C_X^{(r)}(0)$ (r th cumulant)

$\Rightarrow C_1(X) = \mathbb{E}X, \quad C_2(X) = Var[X]$.

* $C_{\frac{X-\mu}{\sigma}}(t) = \log \mathbb{E}(e^{(tX - t\mu)/\sigma})$

$= -\frac{\mu}{\sigma} t + C_X(\frac{t}{\sigma})$

$\therefore$ for $C_1(\frac{X-\mu}{\sigma}) = 0, \quad C_2(\frac{X-\mu}{\sigma}) = 1. \quad C_r(\frac{X-\mu}{\sigma}) = C_r(X)/\sigma^r$ $(r \geq 3)$

* when $\mu = 0, \sigma^2 = 1$ $\quad C_3(Z) = m_3(Z). \quad C_4(Z) = m_4(Z) - 3$

$X \xrightarrow[\text{normalize}]{} Z = \frac{X-\mu}{\sigma} \Rightarrow$ $C_3(Z)$: skewness of X

$C_4(Z)$: curtosis of X

- taylor expand $M_X(t) \Rightarrow$ find $\mathbb{E}X^k$
- show $M_X = M_Y$ to conclude $X \overset{d}{\equiv} Y$

(ex) X, Y are indep $\Rightarrow \mathbb{E}e^{t(X+Y)} = \mathbb{E}e^{tX}\mathbb{E}e^{tY} = M_X(t)M_Y(t)$

if $M_X(t)M_Y(t) = M_Z(t)$, we conclude $X+Y \equiv Z$

* Binomial / Bernoulli

- $X \sim \text{Bern}(p)$ $\quad \mathbb{P}(X=1) = p, \quad \mathbb{P}(X=0) = 1-p$

$\Rightarrow \mathbb{E}X = p\cdot 1 + (1-p)\cdot 0 = p$, $\mathbb{E}X^2 = p\cdot 1^2 + (1-p)\cdot 0^2 = p \Rightarrow \text{Var}X = p(1-p)$

$M_X(t) = \mathbb{E}e^{tX} = pe^t + (1-p)$

- $X \sim B(n,p)$ $\quad \mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ $\quad (k = 0,1,2,\cdots,n)$

$$\mathbb{E}X = \sum_{\underset{\uparrow}{\underset{\sim}{k=0}}}^{n} \underbrace{k\binom{n}{k}}_{= n\binom{n-1}{k-1}} p^k (1-p)^{n-k} = \sum_{l=0}^{n-1} n\binom{n-1}{l} p^l \, p \,(1-p)^{(n-1)-l}$$

1

$$= np\,(p + (1-p))^{n-1} = np$$

note that $X \equiv Z_1 + \cdots + Z_n$ when $Z_i \overset{iid}{\sim} \text{Bern}(p)$

$M_X(t) = (M_Z(t))^n = (pe^t + (1-p))^n$

- $X \sim \text{multi}(n, \mathbb{P})$ $\quad \mathbb{P} = (P_1, \cdots, P_k)^t$ with $P_1 + \cdots + P_k = 1$ $\quad (k \geq 3)$

$= (X_1, \cdots, X_k)$

$$\left[\begin{array}{l} \mathbb{E}X_l = nP_l, \quad \text{Var}(X_l) = nP_l(1-P_l) \quad \text{Cov}(X_l, X_m) = -nP_lP_m \\ \text{mgf}_X(t_1, \cdots, t_k) \left(= \mathbb{E}e^{t_1X_1 + \cdots + t_kX_k}\right) = (P_1e^{t_1} + \cdots + P_ke^{t_k})^n \end{array}\right]$$

$X \overset{d}{\equiv} Z_1 + \cdots + Z_n$ $\quad Z_i \overset{iid}{\sim} \text{multi}(1, \mathbb{P})$

$Z_1 = (Z_{11}, \cdots, Z_{1k})$

$\mathbb{E}Z_{1l} = P_l \quad \mathbb{E}Z_{1l}Z_{1m} = 0$, $\quad M_Z(t) = P_1e^{t_1} + \cdots + P_ke^{t_k}$

$$\mathbb{E}Z_1 = \begin{pmatrix} P_1 \\ \vdots \\ P_k \end{pmatrix} \qquad \text{Var}Z_1 = \begin{pmatrix} P_1(1-P_1) & & -P_lP_m \\ & \ddots & \\ -P_lP_m & & P_k(1-P_k) \end{pmatrix}$$

* Poisson process

Let N_t be number of certain event occurs between 0 and t (time). If following conditions are satisfied, we call $\{N_t \mid t \geq 0\}$ a Poisson Process with occurrence rate λ

(a) Stationarity $\quad N_t \equiv N_{s+t} - N_s \quad , N_0 = 0$

(b) Independent increment $\quad N_t$ and $N_{t+h} - N_t$ is indep for any $h > 0$

(c) Proportionality $\quad P(N_h = 1) = \lambda h + o(h)$ as $h \to 0$

(d) Rareness $\quad P(N_h \geq 2) = o(h)$

Claim. $\quad \mathbb{P}(N_t = k) = e^{-\lambda t} (\lambda t)^k / k!$

Sketch of Pf

$$\mathbb{P}(N_{t+h} = k) = \mathbb{P}(N_{t+h} = k,\ N_t = k) + \mathbb{P}(N_{t+h} = k,\ N_t = k-1) + \mathbb{P}(N_{t+h} = k,\ N_t \leq k-2)$$

$$= \mathbb{P}(N_t = k,\ N_{t+h} - N_t = 0) + \mathbb{P}(N_t = k-1,\ N_{t+h} - N_t = 1) + \text{''} \leq k-2 \quad \text{''} \geq 2)$$

$$= \mathbb{P}(N_t = k)(1 - \lambda h - o(h)) + \mathbb{P}(N_t = k-1)(\lambda h + o(h)) + \mathbb{P}(N_t \leq k-2)\, o(h)$$

$$\therefore \lim_{h \to 0} \frac{\mathbb{P}(N_{t+h} = k) - \mathbb{P}(N_t = k)}{h} = \lambda \left(\mathbb{P}(N_t = k-1) - \mathbb{P}(N_t = k) \right)$$

let $g(k,t) = \mathbb{P}(N_t = k) \cdot e^{\lambda t} / \lambda^k$

$$\Rightarrow \frac{\partial}{\partial t} g(k,t) = g(k-1,t)$$

$g(0,0) = 1, \quad g(1,0) = g(2,0) = \cdots = 0 \qquad \blacksquare$

$f(x) = e^{-\lambda} \lambda^x / x! \qquad x = 0, 1, 2, \cdots \qquad (\lambda > 0)$ is pdf of poisson dist.

$X \sim \text{Poi}(\lambda)$

① $\mathbb{E}e^{tx} = \sum_{x=0}^{\infty} e^{tx-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \cdot \sum_{x=0}^{\infty} (\lambda e^t)^x / x! = e^{\lambda e^t - \lambda}$

② $\mathbb{E}X = \lambda$. $\mathrm{Var}X = \lambda$ ↙ how to use c.g.f

③ $X_1 \sim \mathrm{Pois}(\lambda_1)$. $X_2 \sim \mathrm{Pois}(\lambda_2)$. X_1, X_2 indep

$\Rightarrow M_{X_1+X_2}(t) = M_{X_1}(t)\, M_{X_2}(t) = e^{(\lambda_1+\lambda_2)e^t - (\lambda_1+\lambda_2)}$

$\Rightarrow X_1+X_2 \sim \mathrm{pois}(\lambda_1+\lambda_2)$

* exp / gamma distribution

$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx \quad \rightarrow \quad y = x\beta$

$\Gamma(\alpha) = \int_0^{\infty} \beta^{1-\alpha} y^{\alpha-1} e^{-y/\beta} dy/\beta$

$\therefore \int_0^{\infty} \underbrace{\frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-y/\beta}}_{\text{p.d.f of } \mathrm{Gamma}(\alpha,\beta)} dy = 1$

one can easily prove $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1) \quad \alpha > 1$

(a) $\mathbb{E}X^k = \int_0^{\infty} x^{\alpha+k-1} e^{-x/\beta} \frac{1}{\Gamma(\alpha)\beta^\alpha} dx$

$= \frac{\Gamma(\alpha+k)\beta^{\alpha+k}}{\Gamma(\alpha)\beta^\alpha} \underbrace{\int_0^{\infty} x^{\alpha+k-1} e^{-x/\beta} \frac{1}{\Gamma(\alpha+k)\beta^{\alpha+k}} dx}_{=1}$

$= \alpha(\alpha+1) \cdots (\alpha+k-1)\beta^k$

$\Rightarrow \mathbb{E}X = \alpha\beta \quad \mathrm{Var}X = \alpha\beta^2$

$(\frac{1}{\beta} - t)x = y$

(b) $\mathbb{E}e^{tx} = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^{\infty} x^{\alpha-1} e^{tx - x/\beta} dx$

if $\frac{1}{\beta} - t > 0$

$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot (1/\beta - t)^{-\alpha} \underbrace{\int_0^{\infty} y^{\alpha-1} e^{-y} dy}_{\Gamma(\alpha)}$

$= (1-\beta t)^{-\alpha}$

(c) Similarly $X_i \sim \mathrm{Gam}(\alpha_i, \beta)$, X_1, X_2 indep $\Rightarrow X_1+X_2 \sim \mathrm{Gam}(\alpha_1+\alpha_2, \beta)$

let $\{N_t \mid t \geq 0\}$ be poisson process let W_1 be time til first event.

$$\mathbb{P}(W_1 \leq t) = 1 - \mathbb{P}(N_t = 0) = 1 - e^{-\lambda t}$$

$\Rightarrow$ pdf will be $\lambda e^{-\lambda t}$ $\leftarrow$ exp dist.

$\hookrightarrow$ Gamma$(1, 1/\lambda)$

* Normal dist

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left((x-\mu)^2/\sigma^2\right)}$$

(a)
$$\begin{aligned}\mathbb{E} e^{tX} &= \int e^{tx} \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right) dx \\ &= \int e^{t(\sigma z + \mu)} \phi(z) dz \\ &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-\sigma t)^2} dz}_{=1}\end{aligned}$$

(b) $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ (Similarly)

(c) $\mathbb{E}X = \mu$. $\operatorname{Var} X = \sigma^2$

* Change of variables formula

Thm 1. X is continuous k-dimensional r.v. $u = (u_1, \cdots, u_k)^t : \mathcal{X} \to \mathcal{Y}$

with (a) $\mathbb{P}(X \in \mathcal{X}) = 1$.

(b) $u : \mathcal{X} \to \mathcal{Y}$ is bijective (1-1 correspondence)

(c) $\mathcal{X}$ is open set in $\mathbb{R}^k$. u is continuously differentiable with non-singular Jacobian matrix

$\left(J_u(x) = \left| \frac{\partial}{\partial x_i} u_j(x) \right| \neq 0 \quad \text{for all } x \in \mathcal{X} \right)$

let $Y = u(X)$. Then

$$\mathrm{pdf}_Y(y) = \mathrm{pdf}_X(x) \left| J_{u^{-1}}(y) \right|$$

2. with (a) $\mathbb{P}(X \in \mathcal{X}) = 1$

(b) $u : \mathcal{X} \to \mathcal{Y}$ is $m-1$ correspondence

(c) $\mathcal{X} = \dot{\bigcup}_{r=1}^{m} \mathcal{X}_r$ when each $\mathcal{X}_i$ is open

$u|_{\mathcal{X}_i}$ is 1-1 correspondence with nonsingular Jacobian

let $Y = u(X)$ then

$$\mathrm{pdf}_Y(y) = \sum_{x : u(x) = y} \mathrm{pdf}_X(x) \left|\det\left(\frac{\partial y}{\partial x}\right)\right|^{-1}$$

(ex) location parameter / scale parameter

r.v Z with pdf $f(z)$, $\sigma > 0$, $\mu \in \mathbb{R}$

let $X = \sigma Z + \mu$

then $\mathrm{pdf}_X(x) = f(z) \left|\frac{dx}{dz}\right|^{-1} = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right)$

$N(\underset{\text{loc}}{\mu}, \underset{\text{scale}}{\sigma^2})$, $\mathrm{Gamma}(\alpha, \underset{\text{scale}}{\beta})$

(ex) $Z_1 \sim \mathrm{Gamma}(\alpha_1, \beta)$ $Z_2 \sim \mathrm{Gamma}(\alpha_2, \beta)$, indep

$Z_1 / Z_1 + Z_2 \sim$??

$$\mathrm{pdf}_{Z_1, Z_2}(z_1, z_2) = \Gamma(\alpha_1)^{-1} \Gamma(\alpha_2)^{-1} \beta^{-(\alpha_1 + \alpha_2)} x_1^{\alpha_1 - 1} x_2^{\alpha_2 - 1} e^{-(x_1 + x_2)/\beta} I_{(x_1, x_2 > 0)}$$

$y_1 = z_1 / z_1 + z_2$ $y_2 = z_1 + z_2$

$\mathcal{X} = \{(x_1, x_2) \mid x_1, x_2 > 0\}$ $\mathcal{Y} = \{(y_1, y_2) \mid 0 < y_1 < 1, y_2 > 0\}$

$u^{-1}(y_1, y_2) = (y_1 y_2, (1 - y_1) y_2)'$

$\Rightarrow J_{u^{-1}} = \begin{vmatrix} y_2 & -y_2 \\ y_1 & 1 - y_1 \end{vmatrix} = y_2$

$\Rightarrow \mathrm{pdf}_{Y_1, Y_2}(y_1, y_2) =$ " $(y_1 y_2)^{\alpha_1 - 1} ((1 - y_1) y_2)^{\alpha_2 - 1} e^{-y_2/\beta} |y_2| I_{y_2 > 0} I_{y_1 \in (0,1)}$

$$= \underbrace{\left([\Gamma(\alpha_1+\alpha_2)]^{-1} \beta^{-(\alpha_1+\alpha_2)} y_2^{\alpha_1+\alpha_2-1} e^{-y_2/\beta} I_{(y_2>0)} \right)}_{Y_2 \sim \text{Gamma}(\alpha_1+\alpha_2,\, \beta)} \cdot \underbrace{\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y_1^{\alpha_1-1}(1-y_1)^{\alpha_2-1} I_{(0,1)}(y_1)}_{\text{pdf of Beta}(\alpha_1,\alpha_2)}$$

(ex) $X \sim N(0,1)$ $Y = X^2$

except for $X=0$ (measure 0) it satisfies the condition

$$\text{pdf}_Y(y) = \sum_{x: x^2 = y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \left|\frac{dy}{dx}\right|^{-1} = \underbrace{\frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}}$$

$\hookrightarrow$ Gamma$(1/2, 2)$

If $X_1, \cdots, X_n \overset{iid}{\sim} N(0,1)$

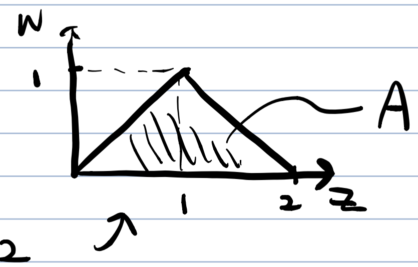
$Y = X_1^2 + \cdots + X_n^2 \sim \text{Gamma}(n/2, 2)$ or $\chi^2(n)$

(ex) $X_{(n)} + X_{(1)}$ for unif $[0,1]$.

$$f_{X_{(1)}, X_{(n)}}(x,y) = n(n-1)(y-x)^{n-2} I(0<x<y<1)$$

$Z = X(1) + X(n)$

$W = X(n) - X(1)$



$$|J_{u^{-1}}| = |(J_u)^{-1}| = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}^{-1} = 1/2$$

$\hookrightarrow$ $0 < Z-W < Z+W < 2$

$$f_{Z,W}(z,w) = n(n-1)w^{n-2} \cdot \frac{1}{2} \cdot I_A(z,w)$$

$$f_Z(z) = \begin{cases} \int_0^z n(n-1) w^{n-2} \frac{1}{2}\, dw = \frac{1}{2} n z^{n-1} \\ \int_0^{2-z} \quad \text{"} \quad dw = \frac{1}{2} n (2-z)^{n-1} \end{cases}$$

$$\text{Var}(Z) = \int_0^2 (z-1)^2 f_Z(z) dz = \int_0^1 (z^2 - 2z + 1) z^{n-1} \frac{n}{2} dz \quad) \text{①}$$

$$+ \int_1^2 (z-1)^2 (2-z)^{n-1} \frac{n}{2} dz \quad) \text{②}$$

① : $\frac{n}{2}\left(\frac{1}{n+2} - \frac{2}{n+1} + \frac{1}{n} \right)$ ② : $\frac{n}{2} \int_0^1 (y-1)^2 \cdot y^{n-1} \frac{n}{2} dy$

$$\Rightarrow\ n\left(\frac{1}{n+2}-\frac{2}{n+1}+\frac{1}{n}\right)=\frac{1}{(n+1)(n+2)}\left\{n(n+1)+(n+1)(n+2)-2n(n+2)\right\}$$

$$=\frac{2}{(n+1)(n+2)}$$

When χ^2-dist appear in Stat.

<u>Thm</u> A is $n\cdot n$ matrix which is symmetric ($A^t=A$), idempotent ($A^2=A$) and $X\sim N(0, I_n)$ $\Rightarrow$ $x^tAx\sim\chi^2_{tr(A)}$

(pf) A is real, symmetric $\Rightarrow$ diagonalizable. $A=UDU^t$ where $UU^t=I_n$.

$D=diag(\lambda_1,\cdots,\lambda_n)$ where λ's are eigenvalues of A

$A v=\lambda v$ $\Rightarrow$ $\lambda v = A v = A^2 v = A(\lambda v) = \lambda(Av) = \lambda^2 v$

$\therefore \lambda = 0$ or 1,

Note that $U^t x \sim N(U^t 0, U^t I_n U) = N(0, I_n)$

$\therefore$ $U^t x = (z_1,\cdots,z_n)'$ where $z_i \overset{iid}{\sim} N(0,1)$

$x^tAx = (U^tx)^t D (U^tx) = \sum \lambda_i z_i^2 = \chi^2_{(\#\text{ of }\lambda_i=1)}$

$tr(A) = tr(UDU^t) = tr(DU^tU) = tr(D) = \#\text{ of }\lambda_i=1$

(ex) $S^2 = \sum (X_i-\bar{X})^2/n-1$

$(n-1)S^2/\sigma^2$ is known to follow $\chi^2(n-1)$

Choose $A = I_n - \frac{1}{n}\mathbb{1}\mathbb{1}^t = I_n - \frac{1}{n}\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & 1 & \cdots & 1\\ \vdots & & & \vdots\\ 1 & 1 & \cdots & 1\end{pmatrix}$

$\blacksquare$