

Outline

- 1) Testing with nuisance parameters
- 2) UMPu multivariate tests
- 3) Conditioning on null sufficient stat

Nuisance Parameters

Common setup: Extra unknown parameters
which are not of direct interest

$$\mathcal{P} = \{P_{\theta, \lambda} : (\theta, \lambda) \in \Omega\}, H_0: \theta \in \Theta_0 \text{ vs } H_1: \theta \in \Theta_1$$

θ parameter of interest

λ nuisance parameter

Issue: λ unknown but might affect
type I error or power of a given test

Ex $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2) \quad Y_1, \dots, Y_m \stackrel{iid}{\sim} N(\nu, \sigma^2)$

μ, ν, σ^2 unknown

$$H_0: \mu = \nu \quad \text{vs} \quad H_1: \mu \neq \nu$$

$$\theta = \mu - \nu \quad \lambda = (\mu + \nu, \sigma^2) \quad \text{or} \quad (\mu, \sigma^2)$$

Ex $X_1 \sim \text{Binom}(n_1, \pi_1) \quad X_2 \sim \text{Binom}(n_2, \pi_2)$

n_1, n_2 known $\Rightarrow$ not nuisance parameters

$$H_0: \pi_1 \leq \pi_2 \quad \text{vs} \quad H_1: \pi_1 > \pi_2$$

Multiparameter Exp. Families

Assume $X \sim p_{\theta, \lambda}(x) = e^{\theta' T(x) + \lambda' u(x) - A(\theta, \lambda)} h(x)$

$\theta \in \mathbb{R}^s$, $\lambda \in \mathbb{R}^r$, both unknown.

How to test $H_0: \theta \in \Theta_0$ vs $H_1: \theta \in \Theta_1$?

Idea: Condition on $U(X)$ to eliminate dep. on λ

1) Sufficiency reduction:

$$(T(X), u(X)) \sim q_{\theta, \lambda}(t, u)$$

$$= e^{\theta' t + \lambda' u - A(\theta, \lambda)} g(t, u)$$

(density wrt $\mu^T \times \mu^u$, e.g. Lebesgue on $\mathbb{R}^{s+r}$)

2) Condition on $U(X)$:

$$q_{\theta}(t | u) = \frac{q_{\theta, \lambda}(t, u)}{\int q_{\theta, \lambda}(z, u) dz}$$

$$= \frac{e^{\theta' t + \cancel{\lambda' u} - \cancel{A(\theta, \lambda)}} g(t, u)}{e^{\theta' z + \cancel{\lambda' u} - \cancel{A(\theta, \lambda)}} g(z, u) dz}$$

$$e^{B_u(\theta)} = \int e^{\theta' z + \cancel{\lambda' u} - \cancel{A(\theta, \lambda)}} g(z, u) dz$$

$$= e^{\theta' t - B_u(\theta)} g(t, u)$$

3) Conditional test:

Test $H_0: \theta \in \Theta_0$ vs. $H_1: \theta \in \Theta_1$ in

s -parameter model $\mathcal{Q}_n = \{q_\theta(t|u) : \theta \in \Theta\}$

Note if $s=1$, this family has MLR in T

Even if $s > 1$, we still have gotten rid of λ

Theorem Let $\mathcal{P}$ be full rank exp. fam. with densities $p_{\theta, \lambda}(x) = e^{\theta^T(x) + \lambda^T U(x) - A(\theta, \lambda)} h(x)$

$\theta \in \mathbb{R}$, $\lambda \in \mathbb{R}^r$, $(\theta, \lambda) \in \Omega$ open, θ_0 possible

a) To test $H_0: \theta \leq \theta_0$ vs. $H_1: \theta > \theta_0$, there is a UMPU test $\phi^*(x) = \gamma(T(x); U(x))$ where

$$\gamma(t; u) = \begin{cases} 1 & t > c(u) \\ \gamma(u) & t = c(u) \\ 0 & t < c(u) \end{cases}$$

with $c(u)$, $\gamma(u)$ chosen to make

$$E_{\theta_0}[\phi^*(x) | U(x)=u] = \alpha$$

b) To test $H_0: \theta = \theta_0$ vs. $H_1: \theta \neq \theta_0$ there is a UMPU test $\phi^*(x) = \gamma(T(x); U(x))$ where

$$\gamma(t; u) = \begin{cases} 1 & t < c_1(u) \text{ or } t > c_2(u) \\ \gamma_i(u) & t = c_i(u) \\ 0 & t \in (c_1(u), c_2(u)) \end{cases}$$

with $c_i(u)$, $\gamma_i(u)$ chosen to make

$$E_{\theta_0}[\phi^*(x) | U(x)=u] = \alpha$$

$$E_{\theta_0}[T(x)(\phi^*(x) - \alpha) | U(x)=u] = 0$$

[Note λ has disappeared from the problem.]

$$\underline{\text{Ex}} : X_i \stackrel{\text{ind.}}{\sim} \text{Pois}(\mu_i) \quad i=1, 2$$

$$H_0: \mu_1 \leq \mu_2 \quad \text{vs.} \quad H_1: \mu_1 > \mu_2$$

$$p_{\mu}(x) = \prod_{i=1}^2 \frac{\mu_i^{x_i} e^{-\mu_i}}{x_i!}$$

$$= e^{X_1 \eta_1 + X_2 \eta_2 - (e^{\eta_1} + e^{\eta_2})} \frac{1}{x_1! x_2!}$$

$$(\text{Where } \eta_i = \log \mu_i \quad H_0: \eta_1 \leq \eta_2 \quad H_1: \eta_1 > \eta_2)$$

$$= e^{X_1 \underbrace{(\eta_1 - \eta_2)}_{\theta} + (X_1 + X_2) \underbrace{\eta_2}_{\lambda} - A(\eta)} \frac{1}{x_1! x_2!}$$

$$H_0: \theta \leq 0 \quad \text{vs.} \quad H_1: \theta > 0$$

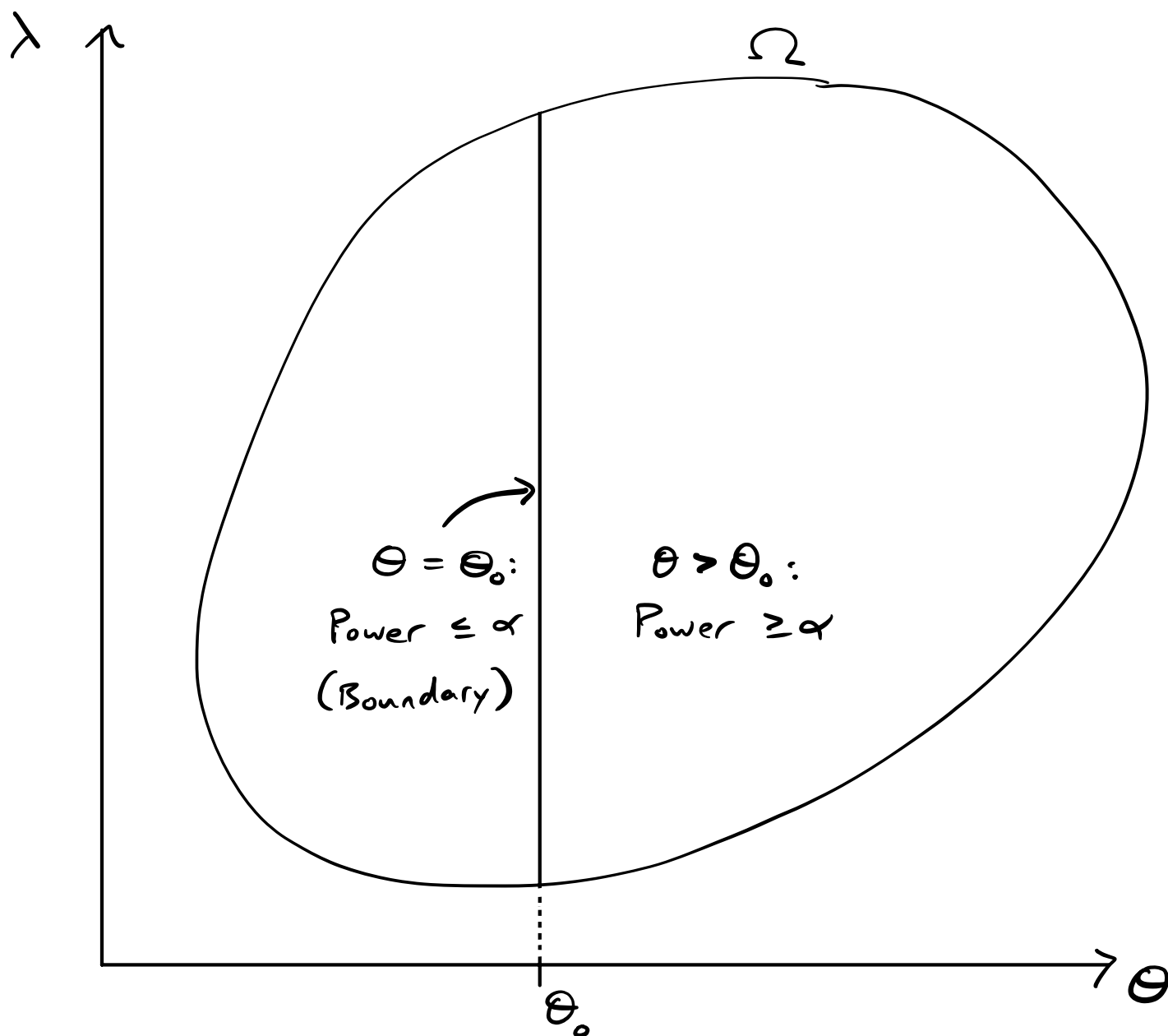
Reject for conditionally large values of X_1 , given $X_1 + X_2 = u$

$$\begin{aligned} P_{\theta}(X_1 = x_1 | U = u) &= e^{x_1 \theta + \cancel{u\lambda} - \cancel{A(\cdot)}} \cdot \frac{1}{x_1! (u-x_1)!} \bigg/ \sum_{x_1=0}^u \frac{1}{x_1! (u-x_1)!} e^{x_1 \theta} \\ &\propto_x e^{x_1 \theta} \cdot \frac{u!}{x_1! (u-x_1)!} \\ &= \text{Binom}(u, \frac{e^{\theta}}{1+e^{\theta}}) \quad e^{\theta} = \mu_1/\mu_2 \end{aligned}$$

$$= \text{Binom}(u, \frac{\mu_1}{\mu_1 + \mu_2})$$

So in the end we do a Binomial test.

Proof Sketch



- 1) Any unbiased test has $\beta(\theta_0, \lambda) = \alpha \quad \forall \lambda$
(continuity)
- 2) Power $\equiv \alpha$ on boundary $\Rightarrow \mathbb{E}_{\theta_0}[\phi | U] \stackrel{q.s.}{=} \alpha$
($U(X)$ complete sufficient on boundary submodel)
- 3) ϕ^* optimal among all tests with conditional level α
(by reduction to univariate model)

Proof Assume ϕ any unbiased test

Step 1: $\mathbb{E}_{\theta, \lambda} |\phi(x)| \leq 1 < \infty \quad \forall (\theta, \lambda) \in \Omega$

(Keener Thm 2.4)

$\Rightarrow \mathbb{E}_{\theta, \lambda} \phi(x)$ infinitely diff. on Ω , can diff. under $\int$

ϕ unbiased $\Rightarrow \mathbb{E}_{\theta, \lambda} [\phi(x)] = \alpha \quad \forall (\theta, \lambda) \in \Omega$

Step 2: Boundary submodel: $\mathcal{P}_{\theta_0} = \{P_{\theta_0, \lambda} : (\theta_0, \lambda) \in \Omega\}$

$$P_{\theta_0, \lambda}(x) = e^{\lambda' u(x) - A(\theta_0, \lambda)} \cdot \frac{e^{\theta_0' T(x)}}{h(x)}$$

$\mathcal{P}_{\theta_0}$ is full-rank, s-param exp. fam, $u(x)$ comp. suff.

Let $f(u) = \mathbb{E}_{\theta_0} [\phi(x) | u(x) = u] - \alpha$

$$\mathbb{E}_{\theta_0, \lambda} [f(u(x))] = \mathbb{E}_{\theta_0, \lambda} [\phi(x)] - \alpha = 0 \quad \forall \lambda$$

$$\Rightarrow f(u) \stackrel{a.s.}{=} 0$$

$$\Rightarrow \mathbb{E}_{\theta_0} [\phi(x) | u(x) = u] = \alpha \quad \forall u$$

Two-sided case: $g(u) = \frac{d}{d\theta} \mathbb{E}_{\theta_0} [\phi | u = u]$

$$= \mathbb{E}_{\theta_0} [(T - \mathbb{E}_{\theta_0} [T | u]) \phi | u]$$

$$= \mathbb{E}_{\theta_0} [T(\phi - \alpha) | u]$$

$$\mathbb{E}_{\theta_0, \lambda} g(u) = \mathbb{E}_{\theta_0, \lambda} [T(\phi - \alpha)] = \frac{\partial}{\partial \theta} B_{\phi}(\theta_0) = 0 \quad \forall \lambda$$

$$\Rightarrow \frac{d}{d\theta} \mathbb{E}_{\theta_0} [\phi | u] \stackrel{a.s.}{=} 0 \quad (\text{Cond'l power has derivative 0 at } \theta_0)$$

Step 3: For any value u , the conditional model is
 $q_\theta(t|u) = e^{\theta t - B_u(\theta)} g(t, u)$, 1-param. exp. fam

In one- / two-sided case, we have shown
 $\psi(t; u)$ is UMP / UMPU in $\mathcal{Q}_u$

Let $\bar{\phi}(t; u) = \mathbb{E}[\phi(x) | T(x)=t, u(x)=u]$

$$\begin{aligned}\mathbb{E}_\theta[\bar{\phi}(T; u) | u=u] &= \mathbb{E}_\theta[\phi(x) | u(x)=u] \\ &= \alpha \text{ if } \theta = \theta_0\end{aligned}$$

$\Rightarrow \bar{\phi}(\cdot; u)$ is a (cond'l) test of H_0 vs. H_1
in $\mathcal{Q}_u$ with power = α at boundary

One-sided case: (or $\theta \leq \theta_0$)
 $\psi(t; u)$ is the UMP test of $\theta = \theta_0$ vs $\theta > \theta_0$
in $\mathcal{Q}_u$, which is a 1-param. exp. fam.

Two-sided case:

$\psi(t; u)$ is the UMP test of $\theta = \theta_0$ vs. $\theta \neq \theta_0$
among tests with power = α , $\frac{d}{d\theta} \text{power} = 0$ @ θ_0
(Keener Thm 12.22, main thm. for two-sided tests)

In either case ψ has higher cond. power
than $\bar{\phi}$, a.s.

For $(\theta, \lambda) \in \Omega_1$:

$$\begin{aligned}\mathbb{E}_{\theta, \lambda}[\phi(x)] &= \mathbb{E}_{\theta, \lambda} \left[\mathbb{E}_{\theta} [\bar{\phi}(\tau; u) \mid u] \right] \\ &\leq \mathbb{E}_{\theta, \lambda} \left[\mathbb{E}_{\theta} [\psi(\tau; u) \mid u] \right] \\ &= \mathbb{E}_{\theta, \lambda}[\phi^*(x)]\end{aligned}$$

$$\underline{E}_X \quad X_1, \dots, X_n \stackrel{\text{iid}}{\sim} N(\mu, \sigma^2) \quad \sigma^2 > 0 \text{ unknown}$$

$$H_0: \mu = 0 \text{ vs. } H_1: \mu \neq 0$$

$$p_{\mu, \sigma^2}(x) = e^{\underbrace{\frac{\mu}{\sigma^2}}_0 \underbrace{\sum X_i}_T - \underbrace{\frac{1}{2\sigma^2}}_{\lambda} \underbrace{\sum X_i^2}_u - \frac{n\mu}{2\sigma^2}} \cdot \left(\frac{1}{2\pi\sigma^2}\right)^{n/2}$$

Condition on $U(X) = \|X\|^2$:

If $\mu = 0$, p is rotationally symmetric

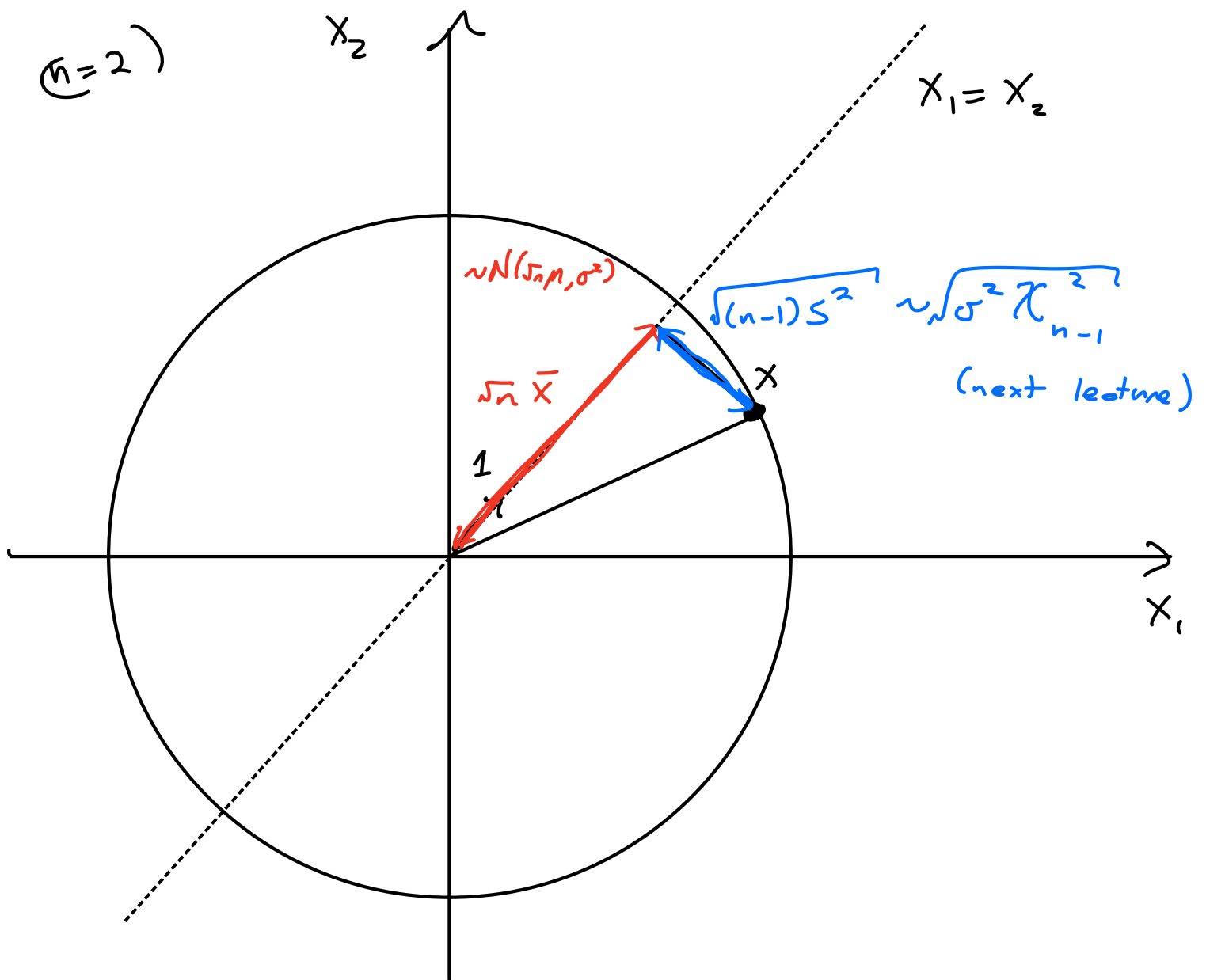
$$\Rightarrow X / \|X\|^2 \stackrel{H_0}{\sim} \text{Unif}(\sqrt{n} \cdot S^{n-1})$$

$$\left(\Leftrightarrow \frac{X}{\|X\|} \stackrel{H_0}{\sim} \text{Unif}(S^{n-1}), \text{ indep. of } \|X\| \right)$$

Optimal test rejects when $\sum X_i$ is extreme given $\|X\|$

$$\begin{aligned} \text{Let } S^2 &= \frac{1}{n-1} \sum (X_i - \bar{X})^2 \\ &= \frac{1}{n-1} \left(\sum X_i^2 - 2\bar{X} \sum X_i + n\bar{X}^2 \right) \\ &= \frac{1}{n-1} \left(\|X\|^2 - \underbrace{n\bar{X}^2}_{\substack{= \frac{1}{n} \sum X_i^2 \\ = \frac{1}{n} \mathbf{1}_n' X}} \right) \end{aligned}$$

$$\begin{aligned} (n-1)S^2 &= \|X\|^2 - \left(\frac{1}{\sqrt{n}} \mathbf{1}_n' X \right)^2 \\ &= \| \text{Proj}_{\mathbf{1}_n^\perp} X \|^2 \end{aligned}$$



$\Leftrightarrow$ rej. when $\frac{\sqrt{n} \bar{X}}{\sqrt{\|x\|^2 - n\bar{X}^2}} = \frac{\sqrt{n} \bar{X} / \|x\|}{\sqrt{1 - \frac{n\bar{X}^2}{\|x\|^2}}}$

is extreme given ~~$\|x\|$~~

$\frac{x}{\|x\|} \perp \|x\|$
under H_0

$\Leftrightarrow$ rej. when $\frac{\sqrt{n} \bar{X}}{\sqrt{S^2}}$ is extreme

$\frac{\sqrt{n} \bar{X}}{\sqrt{S^2}} \stackrel{H_0}{\sim} t_{n-1}$ (next topic)

Permutation Tests

Even if we don't get a UMPU test at the end, conditioning on null suff. stat. still helps.

Ex. $X_1, \dots, X_n \stackrel{iid}{\sim} P$ $Y_1, \dots, Y_m \stackrel{iid}{\sim} Q$ $H_0: P=Q$ $H_1: P \neq Q$

Under H_0 , $P=Q$, $X_1, \dots, X_n, Y_1, \dots, Y_m \stackrel{iid}{\sim} P$

Let $(Z_1, \dots, Z_{n+m}) = (X_1, \dots, X_n, Y_1, \dots, Y_m)$

Under H_0 , $U(Z) = (Z_{(1)}, \dots, Z_{(n+m)})$ compl. suff.

Let $S_{n+m} = \{\text{Permutations on } n+m \text{ elements}\}$

$(X, Y) \mid U \stackrel{H_0}{\sim} \text{Unif}(\{\pi U : \pi \in S_{n+m}\})$

Thus, for any test stat T , if $P=Q$,

$$P_{P,Q}(T(Z) \geq t \mid U) = \frac{1}{(n+m)!} \sum_{\pi \in S_{n+m}} 1\{T(\pi Z) \geq t\}$$

Monte Carlo test: In practice, we sample

$$\pi_1, \dots, \pi_B \stackrel{iid}{\sim} S_{n+m}, \quad \text{e.g. } B = 1000$$

Then $Z, \pi_1 Z, \dots, \pi_B Z \stackrel{iid}{\sim} \text{Unif}(S_{n+m} U)$ under H_0

$$\text{MC } p\text{-value } p = \frac{1}{1+B} \sum_{b=1}^B 1\{T(Z) \leq T(\pi_b Z)\}$$

$$\stackrel{H_0}{\sim} \text{Unif}\left(\left\{\frac{1}{1+B}, \dots, \frac{B-1}{1+B}, 1\right\}\right) \quad (\text{if no ties})$$

$$(p \geq \text{Unif}(\cdot) \quad \text{if there are ties})$$