Testing with one real parameter

Outline

1) Uniformly most powerful test
2) Two-tailed tests
Uniformly most powerful tests

General setup: \( \mathcal{P}, \Theta_0, \Theta_1 \)

**Def** If \( \phi^*(x) \) has sig. level \( \alpha \), and for any other level-\( \alpha \) test \( \phi \) we have
\[
E_{\Theta} \phi^* \geq E_{\Theta} \phi \quad \forall \Theta \in \Theta_1,
\]
then \( \phi^* \) is **uniformly most powerful (UMP)**

Typically only exist for 1-sided testing in certain 1-parameter families.

**Def** A model \( \mathcal{P} \) is **identifiable** if
\[
\Theta_1 \neq \Theta_2 \implies P_{\Theta_1} \neq P_{\Theta_2} \quad (\exists A: P_{\Theta_1}(A) \neq P_{\Theta_2}(A))
\]

**Def** Assume \( \mathcal{P} = \{ P_\Theta : \Theta \in \Theta \subseteq \mathbb{R}^3 \} \) has densities \( P_\Theta \), and is identifiable. We say \( \mathcal{P} \) has **monotone likelihood ratios (MLR)** if there is some statistic \( T(X) \) s.t.
\[
\frac{P_{\Theta_2}(X)}{P_{\Theta_1}(X)} \text{ is a nondecreasing function of } T(X),
\]
for any \( \Theta_1 < \Theta_2 \) [same \( T(X) \) for all \( \Theta \)'s] \(
\left( \frac{c}{c_o} = \infty \text{ if } c > 0, \quad \frac{c}{c_o} \text{ undefined} \right)
\)

**Ex.** Exp. fam: \( e^{(x_i - \gamma_0) \Sigma T(X) - v(A_{\gamma_1}) - A(\gamma_0))} \uparrow \in \Theta T(x_i) \)
Theorem  Assume \( \mathcal{P} \) has MLE, test \( H_0: \theta = \theta_0 \) vs \( H_1: \theta > \theta_0 \) at level \( \alpha = (0, 1) \)

Let \( \phi^*(x) = \begin{cases} 0 & T(x) < c \\ \gamma & T(x) = c \\ 1 & T(x) > c \end{cases} \)

with \( c, \gamma \) chosen so \( \mathbb{E}_{\theta_0} \phi^*(x) = \alpha \in (0, 1) \)

a) \( \phi^* \) is a UMP level-\( \alpha \) test

b) If \( \theta_1 < \theta_0 \) then \( \phi^* \) minimizes \( \mathbb{E}_{\theta_1} \phi(x) \)

among all tests with \( \mathbb{E}_{\theta_0} \phi(x) = \alpha \)

Proof

a) Suppose \( \theta_1 > \theta_0 \) and \( \phi \) has level \( \leq \alpha \)

\[
\mathbb{E}_{\theta_1} \phi^*(x) = \mathbb{E}_{\theta_0} \phi(x) \text{ since } \phi^* \text{ is a LRT for } H_0: \theta = \theta_0 \text{ vs. } H_1: \theta = \theta_1
\]

c) \( \theta_1 < \theta_0 \), assume \( \mathbb{E}_{\theta_0} \tilde{\phi}(x) = \mathbb{E}_{\theta_0} \phi^*(x) = \alpha \)

Both \( 1-\phi^* \) and \( 1-\tilde{\phi} \) are tests of \( H_0: \theta = \theta_0 \) vs \( H_1: \theta = \theta_1 \), both have sig. level \( 1-\alpha \)

\( 1-\phi^* \) is a LRT since \( \frac{L_1}{L_0}(x) \) is non-incre. in \( T(x) \)

\[
\Rightarrow \quad \mathbb{E}_1(1-\phi) \leq \mathbb{E}_{\theta_1}(1-\phi^*) = 1-\alpha \quad \blacksquare
\]

Intuition \( \phi^* \) is a LRT for \( H_0: \theta = \theta_0 \) vs \( H_1: \theta = \theta_1 \)

for any pair \( \theta_0 < \theta_1 \) (sig. level depends on \( \theta_0 \))
UMP test: Picture

Best on $H_0$ (among tests w. exactly level $\alpha$)

Best on $H_1$ (among tests w. level $\leq \alpha$)

$\beta(\theta)$

$\alpha$

$H_0$

$\Theta_0$

$H_1$
One-sided tests in general

\[ P = \{ \mathcal{P}_\theta : \theta \in \Theta \subset \mathbb{R}^2, \ \theta \in \Theta \} \]

\[ H_0 : \theta \leq \theta_0 \text{ vs } H_1 : \theta > \theta_0 \text{ called one-sided hypothesis} \]

Often, no UMP test exists

Ex. Laplace: \[ X_1, \ldots, X_n \sim \frac{1}{2} e^{-|x - \Theta|} \]

LRT for \[ H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta = \theta_1 (> \theta_0) \]

\[ \log \left( \frac{p_1(x)}{p_0(x)} \right) = \sum_{i=1}^{n} |X_i - \theta| - |X_i - \theta_1| \]

\[ = \sum T(X_i) \]

\[ T(x) = \begin{cases} 
\theta_0 - \theta_1, & x \leq \theta_0 \\
2x - \theta_0 - \theta_1, & \theta_0 \leq x \leq \theta_1 \\
\theta_1 - \theta_0, & x > \theta_1 
\end{cases} \]

Very dependent on specific values of \( \theta_0 \) and \( \theta \).

Test \[ H_0 : \theta \leq 0 \text{ vs } H_1 : \theta > 0 : \text{ No UMP test} \]

Test \[ H_0 : \theta = 0 \text{ vs } H_1 : \theta = \varepsilon, \ \varepsilon > 0 : \]

\[ \sum T(X_i) = -\varepsilon \# \{ X_i \leq 0 \} + \varepsilon \# \{ X_i \geq \varepsilon \} + \sum_{\varepsilon < \xi \leq 0, \varepsilon} 2X_i - \varepsilon \]

\[ \frac{1}{\varepsilon} \sum T(X_i) \xrightarrow{\varepsilon \to 0} \# \{ X_i > 0 \} - \# \{ X_i \leq 0 \} \ 	ext{Sign test} \]
Stochastically incr.

**Def** A real-valued statistic $T(X)$ is stochastically increasing in $\theta$ if

$$P_\theta(T(X) \leq t)$$

is non-incr. in $\theta$, $\forall t$

If $\phi(x)$ is right-tailed test based on $T(X)$:

$$\phi(x) = 1\{T(x) > c\} + \gamma 1\{T(x) = c\}$$

and $T(X)$ is stochastically increasing in $\theta$,

$$E_\theta \phi(x) = (1-\gamma) P_\theta(T > c) + \gamma P_\theta(T = c) \xrightarrow{??} \theta$$

Ex \( X_i \text{iid } \rho(x - \theta) \) (location family)

- $T(X) = \text{sample mean, median, sign statistic}$

Ex \( X_i \text{iid } 1/\theta \rho(x/\theta) \) (scale family)

- $T(X) = \sum X_i^2$ or $\text{median}(1X_1, \ldots, 1X_n)$
Two-sided Alternatives

Setup: \( S = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R} \}, \theta_0 \in \Theta' \)

Test \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta \neq \theta_0 \)

(Can be generalized naturally to \( H_0 : \theta \in [\theta_1, \theta_2] \))
Two-tailed test rejects when $T(X)$ is “extreme”

$$\phi(x) = \begin{cases} 
1 & T(X) > c_2 \text{ or } T(X) < c_1 \\
0 & T(X) \in (c_1, c_2) \\
y_i & T(X) = c_i 
\end{cases}$$

Let $\alpha_1 = P_{\theta_0} (T < c_1) + y_1 P_{\theta_0} (T = c_1)$

$\alpha_2$ similar for upper tail

Need $\alpha_1 + \alpha_2 = \alpha$, but how to balance?

Idea 1: Equal-tailed test: $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$

Def $\phi(x)$ is unbiased if $\inf_{\theta \in \Theta} E_\theta \phi(x) \geq \alpha$

Idea 2: Unbiased test: ensure $\min \beta_\phi (\theta) = \alpha$

(usually $\iff \frac{d\beta_\phi (\theta_0)}{d\theta} = 0$)
Theorem Assume $X_i \sim e^{\theta T(x)} h(x)$

$H_0 : \theta \in [\theta_1, \theta_2]$ vs $H_1 : \theta < \theta_1$ or $\theta > \theta_2$

(possibly $\theta_1 = \theta_2$)

Then

a) The unbiased test based on $\Sigma T(X_i)$ with sig. level $= \alpha$ is UMP among all unbiased tests (UMPu)

b) If $\theta_1 < \theta_2$, the UMPu test can be found by solving for $c_i, \chi_i$ s.t. $E_{\theta_1} \phi = E_{\theta_2} \phi = \alpha$

c) If $\theta_1 = \theta_2 = \theta_0$, the UMPu test can be found by solving for $c_i, \chi_i$ s.t. $E_{\theta_0} \phi(x) = \alpha$ and

$$\frac{d \beta_{\phi}(\theta)}{d \theta}(\theta_0) = E_{\theta_0} \left[ \Sigma T(x_i)(\phi(x) - \alpha) \right] = 0$$

(Proof in Keener)