Testing with one real parameter

10/12/2023

Outline

1) Uniformly most powerful test
2) Two-tailed tests

Uniformly most powerful tests

General setup: $\mathcal{P}$, Θ_0, Θ_1

Def If $\phi^*(x)$ has sig. level α, and for any other level-α test ϕ we have

$$\mathbb{E}_\theta \phi^* \geq \mathbb{E}_\theta \phi \qquad \forall \theta \in \Theta,$$

then ϕ^* is <u>uniformly most powerful</u> (UMP)

Typically only exist for 1-sided testing in certain 1-parameter families.

Def A model $\mathcal{P}$ is <u>identifiable</u> if

$$\theta_1 \neq \theta_2 \Rightarrow P_{\theta_1} \neq P_{\theta_2} \quad (\exists A : P_{\theta_1}(A) \neq P_{\theta_2}(A))$$

Def Assume $\mathcal{P} = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$ has densities p_θ, and is identifiable. We say $\mathcal{P}$ has <u>monotone likelihood ratios</u> (MLR) if there is some statistic $T(X)$ s.t.

$\dfrac{p_{\theta_2}}{p_{\theta_1}}(X)$ is a nondecreasing function of $T(X)$, for any $\theta_1 < \theta_2$ [same $T(X)$ for all θ's]

$\left(\frac{c}{0} = \infty \text{ if } c > 0, \ \frac{0}{0} \text{ undef.}\right)$

Ex. Exp. fam: $e^{(\eta_1 - \eta_0)\sum T(x_i) - n(A(\eta_1) - A(\eta_0))} \nearrow$ in $\sum T(x_i)$

Theorem Assume $\mathcal{P}$ has MLR, test $H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$ at level $\alpha = (0,1)$

Let $\phi^*(x) = \begin{cases} 0 & T(x) < c \\ \gamma & T(x) = c \\ 1 & T(x) > c \end{cases}$,

with c, γ chosen so $\mathbb{E}_{\theta_0} \phi^*(x) = \alpha \in (0,1)$

a) ϕ^* is a UMP level-α test

b) If $\theta_1 < \theta_0$ then ϕ^* minimizes $\mathbb{E}_{\theta_1} \phi(X)$ among all tests with $\mathbb{E}_{\theta_0} \phi(X) = \alpha$

Proof

a) Suppose $\theta_1 > \theta_0$ and $\tilde{\phi}$ has level $\leq \alpha$

$\Rightarrow \mathbb{E}_{\theta_1} \phi^*(X) \geq \mathbb{E}_{\theta_1} \tilde{\phi}(X)$ since ϕ^* is a LRT for $H_0: \theta = \theta_0$ vs. $H_1: \theta = \theta_1$

c) $\theta_1 < \theta_0$, assume $\mathbb{E}_{\theta_0} \tilde{\phi}(X) = \mathbb{E}_{\theta_0} \phi^*(x) = \alpha$

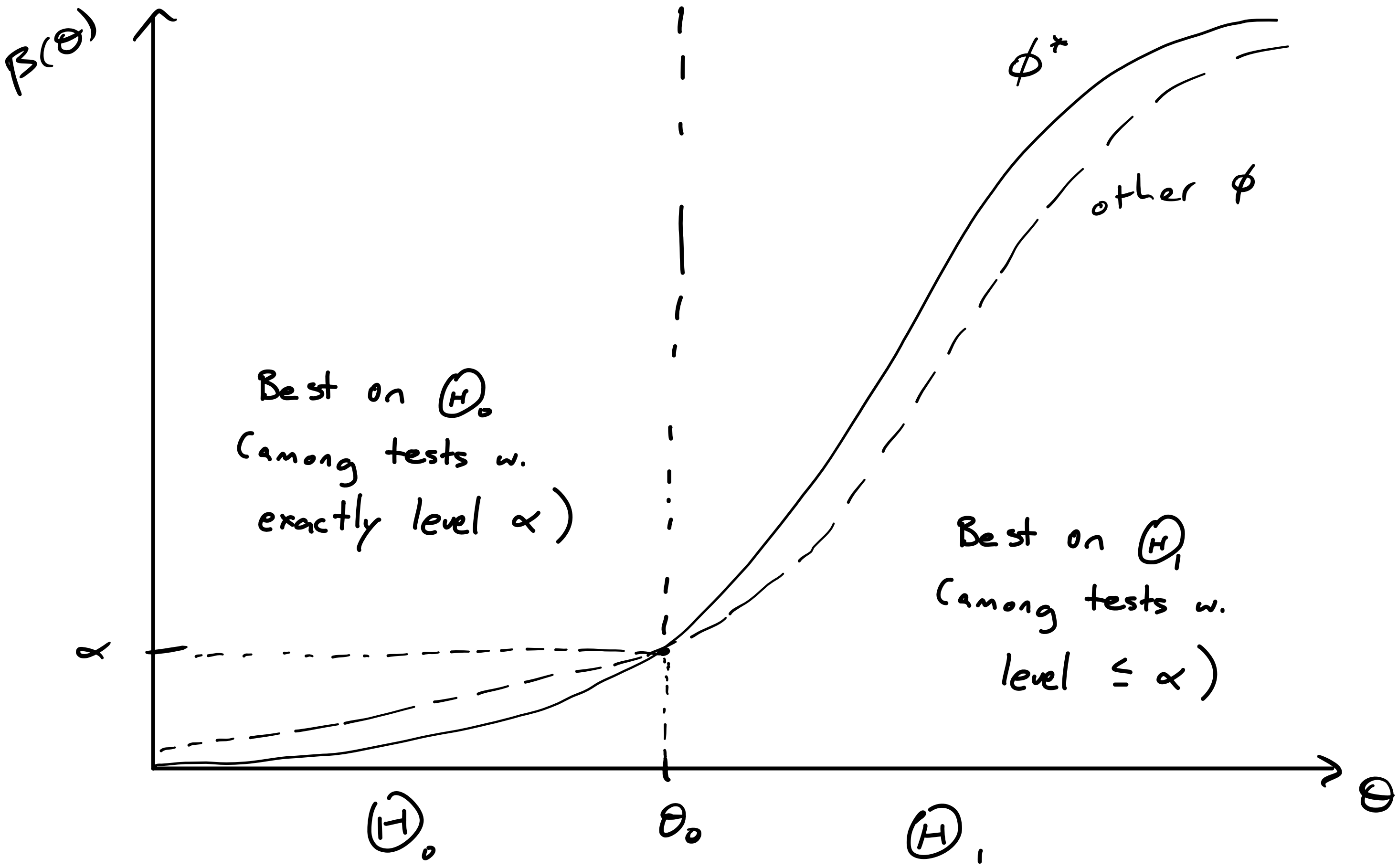
Both $1-\phi^*$, $1-\tilde{\phi}$ are tests of $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$
both have sig. level $1-\alpha$

$1-\phi^*$ is a LRT since $\frac{p_1}{p_0}(X)$ is non-incr. in $T(X)$

$\Rightarrow \mathbb{E}_{\theta_1}(1-\tilde{\phi}) \leq \mathbb{E}_{\theta_1}(1-\phi^*) = 1-\alpha$ $\boxtimes$

Intuition ϕ^* is a LRT for $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$ for <u>any</u> pair $\theta_0 < \theta_1$ (sig. level depends on θ_0)

UMP test: Picture



One-sided tests in general

$\mathcal{P} = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$, $\quad \theta_0 \in \Theta$

$H_0 : \theta \overset{\geq}{\leq} \theta_0$ vs $H_1 : \theta \overset{<}{>} \theta_0$ called <u>one-sided hypothesis</u>

Often, no UMP test exists

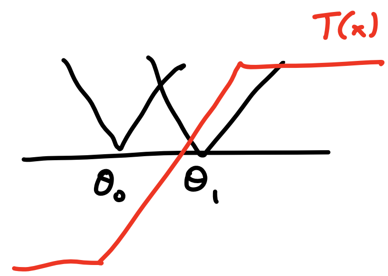
<u>Ex. Laplace:</u> $\quad X_1, \dots, X_n \overset{iid}{\sim} \frac{1}{2} e^{-|x-\theta|}$

LRT for $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1 (> \theta_0)$

$$\log\left(p_1(x) / p_0(x) \right) = \sum_{i=1}^n |x_i - \theta_0| - |x_i - \theta_1|$$

$$= \sum T(x_i)$$

$$T(x) = \begin{cases} \theta_0 - \theta_1 & x \leq \theta_0 \\ 2x - \theta_0 - \theta_1 & \theta_0 \leq x \leq \theta_1 \\ \theta_1 - \theta_0 & x \geq \theta_1 \end{cases}$$



Very dependent on specific values of θ_0 and θ_1

Test $H_0 : \theta \leq 0$ vs $H_1 : \theta > 0$: No UMP test

Test $H_0 : \theta = 0$ vs $H_1 : \theta = \varepsilon$, $\varepsilon \downarrow 0$:

$$\sum T(X_i) = -\varepsilon \#\{X_i \leq 0\} + \varepsilon \#\{X_i \geq \varepsilon\} + \sum_{X_i \in [0,\varepsilon]} 2X_i - \varepsilon$$

$$\frac{1}{\varepsilon} \sum T(X_i) \xrightarrow[\varepsilon \to 0]{} \#\{X_i > 0\} - \#\{X_i \leq 0\} = 2\#\{X_i > 0\} - n$$

$$n + \frac{1}{2\varepsilon} \sum T(X_i) \xrightarrow[\varepsilon \to 0]{} \#\{X_i > 0\} \overset{\theta=0}{\sim} \text{Binom}(n, \tfrac{1}{2})$$ <u>Sign test</u>

Stochastically incr.

Def A real-valued statistic $T(X)$ is stochastically increasing in θ if

$$\mathbb{P}_\theta(T(X) \le t) \text{ is non-incr. in } \theta, \ \forall t$$

If $\phi(x)$ is right-tailed test based on $T(X)$:

$$\phi(x) = 1\{T(x) > c\} + \gamma 1\{T(x) = c\}$$

and $T(X)$ is stochastically increasing in θ,

$$\mathbb{E}_\theta \phi(X) = (1-\gamma)\mathbb{P}_\theta(T > c) + \gamma \mathbb{P}_\theta(T \ge c) \nearrow_{\text{in } \theta}$$

Ex $X_i \overset{iid}{\sim} p(x-\theta)$ (location family)

$T(X)$ = sample mean, median, sign statistic

Ex $X_i \overset{iid}{\sim} \frac{1}{\theta} p(x/\theta)$ (scale family)

$T(x) = \sum X_i^2$ or $\text{median}(|X_1|, \dots, |X_n|)$

Two-sided Alternatives

Setup: $\mathcal{P} = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$, $\theta_0 \in \Theta^\circ$

Test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$

(Can be generalized naturally to $H_0 : \theta \in [\theta_1, \theta_2]$)

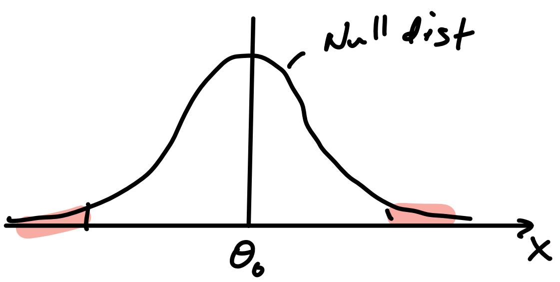
Two-tailed test rejects when $T(X)$ is "extreme"

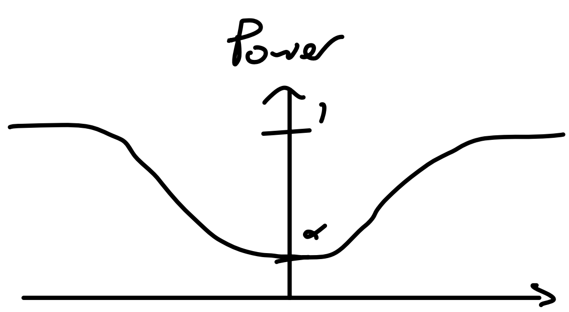
$$\phi(x) = \begin{cases} 1 & T(X) > c_2 \text{ or } T(X) < c_1 \\ 0 & T(X) \in (c_1, c_2) \\ \gamma_i & T(X) = c_i \end{cases}$$

Two ways to reject. How to balance?

For symmetric distributions like $N(\theta, 1)$,
natural choice is to equalize "lobes" of rej. region

$$\phi_2(x) = 1\{|X - \theta_0| > z_{\alpha/2}\} \text{ for } H_0 : \theta = \theta_0$$





For asymmetric dists, or interval null $H_0 : \theta \in [\theta_1, \theta_2]$,
more complicated

Equal-tailed & unbiased tests

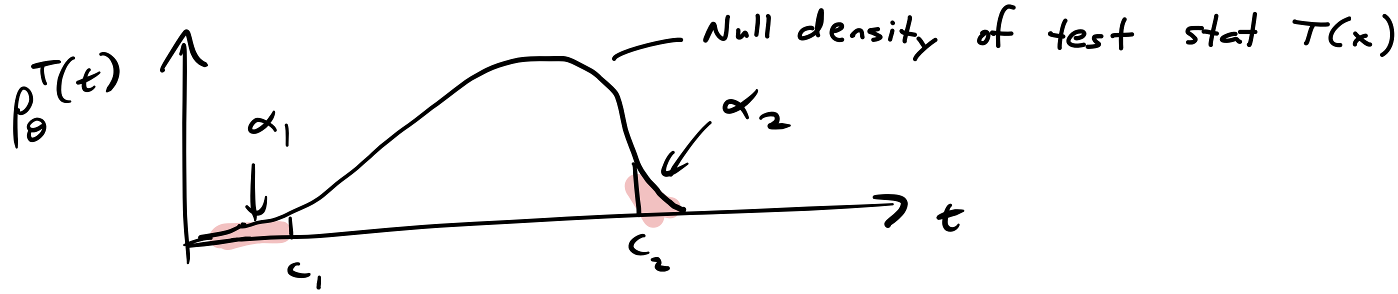
Point null ($H_0 : \theta = \theta_0$)

Let $\alpha_1 = \mathbb{P}_{\theta_0}(T < c_1) + \gamma_1 \mathbb{P}_{\theta_0}(T = c_1)$

$\alpha_2 = \mathbb{P}_{\theta_0}(T > c_2) + \gamma_2 \mathbb{P}_{\theta_0}(T = c_2)$

Valid if $\alpha_1 + \alpha_2 = \alpha$ (α_1 is "free parameter")

Idea 1: Equal-tailed test: $\alpha_1 = \alpha_2 = \frac{\alpha}{2}$



Ex $X \sim \text{Exp}(\theta)$, test $H_0 : \theta = 1$

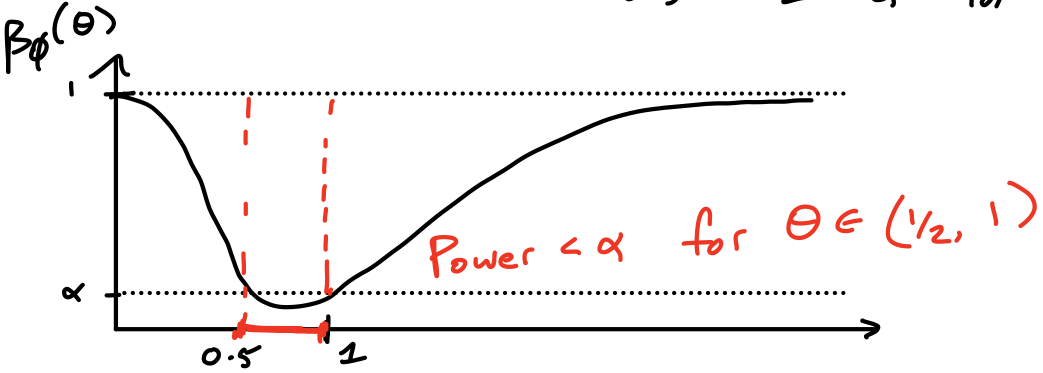
Solve for cutoffs: $\frac{\alpha}{2} = \mathbb{P}_1(X \le c_1) = 1 - e^{-c_1} \Rightarrow c_1 = -\log(1 - \alpha/2)$

$1 - \frac{\alpha}{2} = 1 - e^{-c_2} \Rightarrow c_2 = -\log(\alpha/2)$

$\phi(x) = 1\{X < -\log(1-\alpha/2)\} + 1\{X > -\log(\alpha/2)\}$

$\beta_\phi(\theta) = \mathbb{P}_\theta\left\{\overset{\text{Exp}(1)}{\frac{X}{\theta}} < \frac{-\log(1-\alpha/2)}{\theta}\right\} + \mathbb{P}_\theta\left\{\frac{X}{\theta} > \frac{-\log(\alpha/2)}{\theta}\right\}$

$= 1 - (1-\alpha/2)^{1/\theta} + (\alpha/2)^{1/\theta} = \alpha$ for $\theta = 1$ or $1/2$



Unbiased tests

Def $\phi(X)$ is **unbiased** if $\inf_{\theta \in \Theta_1} \mathbb{E}_\theta \phi(X) \geq \alpha$

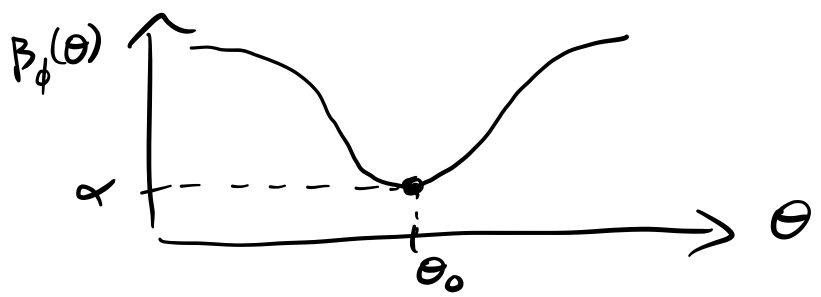
Idea 2: **Unbiased test**: ensure $\min \beta_\phi(\theta) = \alpha$

Choose c_1, γ_1 and c_2, γ_2 to solve:

$$\beta_\phi(\theta_0) = \alpha$$

$$\frac{d\beta_\phi}{d\theta}(\theta_0) = 0$$

(2 equations, "2" unknowns)



Ex: 1-parameter exp. family, $H_0: \eta = \eta_0$ vs $H_1: \eta \neq \eta_0$

$X \sim e^{\eta T(x) - A(\eta)} h(x)$ (MLR in $T(X)$)

Assume $T(X)$ continuous, solve

$$\alpha = \beta_\phi(\eta_0) = \mathbb{P}_{\eta_0}(T < c_1) + \mathbb{P}_{\eta_0}(T > c_2)$$

$$0 = \frac{d\beta_\phi}{d\eta}(\eta_0) = \mathrm{Cov}_{\eta_0}(\phi(T), T)$$

$$= \mathbb{E}_{\eta_0}\left[(\phi(T) - \alpha)\, T(X)\right]$$

Theorem Assume $X_i \overset{iid}{\sim} e^{\theta T(x) - A(\eta)} h(x)$

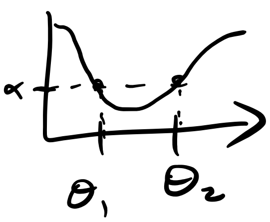
$H_0 : \theta \in [\theta_1, \theta_2]$ vs $H_1 : \theta < \theta_1$ or $\theta > \theta_2$

(possibly $\theta_1 = \theta_2$)

Then

(rejecting for extreme values of)

a) The unbiased test based on $\sum T(X_i)$ with sig. level $= \alpha$ is UMP among all unbiased tests (UMPU)



b) If $\theta_1 < \theta_2$ the UMPU test can be found by solving for c_i, γ_i s.t. $\mathbb{E}_{\theta_1} \phi = \mathbb{E}_{\theta_2} \phi = \alpha$

c) If $\theta_1 = \theta_2 = \theta_0$ the UMPU test can be found by solving for c_i, γ_i s.t. $\mathbb{E}_{\theta_0} \phi(X) = \alpha$ and

$$\frac{d\beta_\phi}{d\theta}(\theta_0) = \mathbb{E}_{\theta_0}\left[\sum T(X_i)(\phi(x) - \alpha)\right] = 0$$

(Proof in Keener)