Lecture 5

Outline

1) Completeness
2) Ancillarity
3) Basu's Theorem

Completeness

Def $T(X)$ is complete for $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$

if $\mathbb{E}_\theta f(T(X)) = 0 \quad \forall \theta$

$$\Rightarrow \quad f(T) \overset{a.s.}{=} 0 \quad \forall \theta$$

"no nontrivial unbiased estimators of 0"

Name comes from a prior notion of

$\mathcal{P}^T = \{P_\theta^T : \theta \in \Theta\}$ being "complete;"

[see HW Prob]

[For now, think "minimal +"]

Ex. (Cont'd) Laplace location family has minimal suff stat. $S = (X_{(i)})_{i=1}^n$. Complete?

No: Let $M(S) = \text{median}(X)$

$$\overline{X}(S) = \frac{1}{n} \sum X_i$$

$$\mathbb{E}_\theta \overline{X} = \mathbb{E}_\theta M = \theta \quad \text{(by symmetry)}$$

$$\mathbb{E}_\theta \left[\overline{X}(S) - M(S) \right] = 0 \quad \forall \theta$$

$S(X)$ still has "a lot of fluff"

Ex $X_1, \ldots, X_n \overset{iid}{\sim} U[0, \theta]$ $\theta \in (0, \infty)$

Can show $T(X) = X_{(n)}$ min. suff. Complete?

Find density of $T(X)$:

$$\mathbb{P}_\theta(T \leq t) = \left(\frac{t}{\theta} \wedge 1\right)^n = \left(\frac{t}{\theta}\right)^n \wedge 1$$

$$\Rightarrow p_\theta(t) = \frac{d}{dt} \mathbb{P}_\theta(T \leq t)$$

$$= n \frac{t^{n-1}}{\theta^n} 1\{t \leq \theta\}$$

Suppose $0 = \mathbb{E}_\theta f(T)$ $\forall \theta > 0$

$$= \frac{n}{\theta^n} \int_0^\theta f(t) t^{n-1} dt \quad \forall \theta > 0$$

$$\Rightarrow \int_0^\theta f(t) t^{n-1} dt = 0 \quad \forall \theta > 0$$

$$\Rightarrow f(t) t^{n-1} = 0 \quad \text{a.e. } t > 0$$

Def Let $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ be an exponential family with densities wrt μ

$$p_\theta(x) = e^{\eta(\theta)' T(x) - B(\theta)} h(x)$$

Assume wlog $\nexists\, \alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^s$ with $\beta' T(x) \overset{a.s.}{=} \alpha$

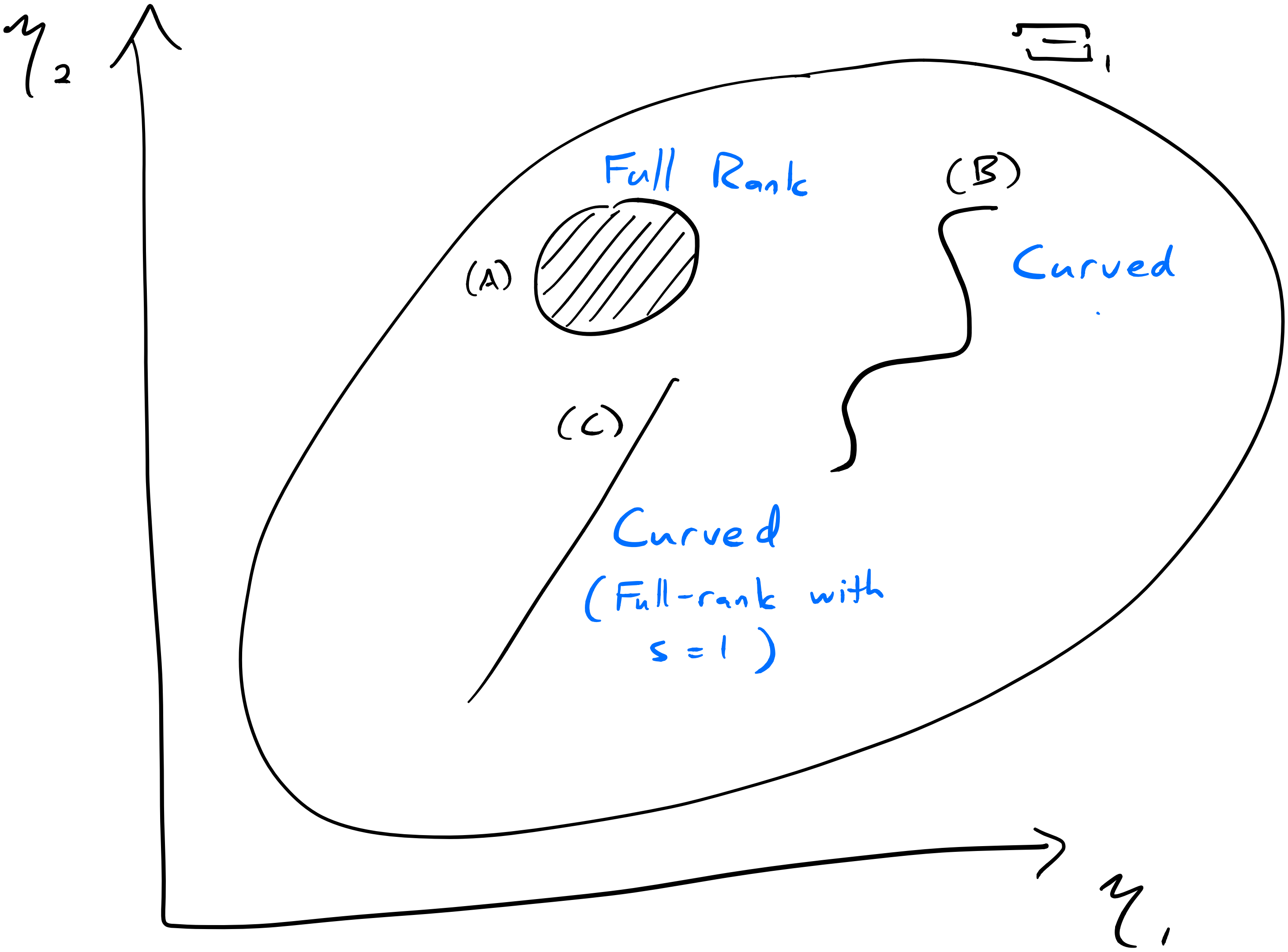
[If so, replace $T(x)$ with a linearly independent basis]

If $\Xi = \eta(\Theta) = \{\eta(\theta) : \theta \in \Theta\}$ contains an open set, we say $\mathcal{P}$ is <u>full-rank</u>.

Otherwise we say $\mathcal{P}$ is <u>curved</u>.

<u>**Theorem**</u> If $\mathcal{P}$ is full rank then $T(X)$ is complete sufficient

Proof in Lehmann & Romano, Thm. 4.3.1



η_2
Ξ_1
Full Rank
(A)
(B)
Curved
(C)
Curved
(Full-rank with
s = 1)
η_1

Theorem If $T(X)$ complete sufficient for $\mathcal{P}$ then $T(X)$ is minimal

Game plan for completeness proofs: show two things are a.s. equal by showing they have expectation.

Proof Assume $S(X)$ is minimal suff (*)

Then $S(X) \stackrel{a.s.}{=} f(T(X))$ (since T suff.)

Note $m(S(X)) = \mathbb{E}_\theta[T(X) \mid S(X)]$ does not depend on θ.

Let $g(t) = t - m(f(t))$

$$\mathbb{E}_\theta[g(T(X))] = \mathbb{E}_\theta T(X) - \mathbb{E}_\theta[m(S(X))]$$

$$= \mathbb{E}_\theta T(X) - \mathbb{E}_\theta[\mathbb{E}[T \mid S]]$$

$$= 0$$

$$\Rightarrow g(T(X)) \stackrel{a.s.}{=} 0 \qquad \text{(completeness)}$$

$$\Rightarrow T(X) \stackrel{a.s.}{=} m(S(X))$$

Ancillarity

Def $V(X)$ is *ancillary* for $\mathcal{P} = \{P_\theta : \theta \in \Theta\}$ if its distribution does not depend on θ. (V carries no info. about θ)

Theorem (Basu)

If $T(X)$ is complete sufficient and $V(X)$ is ancillary for $\mathcal{P}$, then

$$V(X) \perp\!\!\!\perp T(X) \quad \text{for all } \theta \in \Theta$$

Proof Want to show $(\forall\ A, B,\ \forall \theta)$

$$\mathbb{P}_\theta(V \in A, T \in B) = \mathbb{P}_\theta(V \in A)\, \mathbb{P}_\theta(T \in B)$$

equiv: $\mathbb{P}_\theta(V \in A \mid T \in B) = \mathbb{P}_\theta(V \in A)$ $\left(\text{if } \mathbb{P}_\theta(T \in B) > 0\right)$

Let $q_A(T(X)) = \mathbb{P}(V \in A \mid T)$ [suff.]

no θ

$p_A = \mathbb{P}(V \in A)$ [anc.]

no θ

$$\mathbb{E}_\theta[q_A(T) - p_A] = p_A - p_A = 0, \quad \forall \theta$$

$$\Rightarrow q_A(T) \overset{a.s.}{=} p_A \quad \forall \theta$$

$\boxtimes$

<u>Remark</u> Keep in mind while trying to use Basu:

Ancillarity, Completeness, Sufficiency are all properties of a statistic <u>wrt a family $\mathcal{P}$.</u> Independence is a property wrt a particular distribution P_θ

$\Rightarrow$ If T, V don't have the desired properties relative to $\mathcal{P}$, they might have them relative to subfamilies.

<u>Ex.</u> $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ $\mu \in \mathbb{R}$, $\sigma^2 > 0$

Let $\mathcal{Q}_{\sigma^2} = \{ N(\mu, \sigma^2)^n : \mu \in \mathbb{R} \}$

$\bar{X} = \frac{1}{n} \sum X_i$ is complete suff. for $\mathcal{Q}_{\sigma^2}$

$S^2 = \frac{1}{n-1} \sum (X_i - \bar{X})^2$ is ancillary:

Let $Z_i = X_i - \mu \overset{iid}{\sim} N(0, \sigma^2)$ NOT statistics!

$X_i - \bar{X} = Z_i - \bar{Z} \Rightarrow S^2 = \frac{1}{n-1} \sum (Z_i - \bar{Z})^2$

Therefore $\bar{X} \perp\!\!\!\perp S^2$ for all $\mu \in \mathbb{R}$, $\sigma^2 > 0$

Note S^2 is <u>not</u> ancillary in the full family $\mathcal{P} = \{N(\mu, \sigma^2)^n : \mu \in \mathbb{R}, \sigma^2 > 0\}$ so we couldn't have applied Basu without breaking $\mathcal{P}$ up into $\bigcup_{\sigma^2 > 0} \mathcal{Q}_{\sigma^2}$