You may disregard measure-theoretic niceties about conditioning on measure-zero sets, almost-
sure equality vs. actual equality, “all functions” vs. “all measurable functions,” etc. (unless the
problem is explicitly asking about such issues).

If you need to write code to answer a question, show your code. If you need to include a plot,
make sure the plot is readable, with appropriate axis labels and a legend if necessary. Points will
be deducted for very hard-to-read code or plots.

Finally, anytime I ask you to calculate things numerically, it always goes without saying that
Monte Carlo (calculating an expectation by repeatedly sampling data from an appropriate distri-
bution and taking the average) is a valid numerical method. There is no need to ask permission,
but please use good judgment about how many samples to take so that your numerical error is
not too high: if you report a number it should be correct to a couple significant digits; if you are
comparing two numbers, the precision of your calculation should be high enough for the difference
to be meaningful; and plots of smooth functions should appear smooth enough for the plot to be
readable.

1. MLR and location families

(a) Assume $X \sim p_\theta(x) = p_0(x - \theta)$, a location family with $p_0$ continuous and strictly
positive. Show that the family has MLR in $x$ if and only if $\log p_0$ is concave.

(b) Consider testing in the Cauchy location family:

$$p_\theta(x) = \frac{1}{\pi(1 + (x - \theta)^2)}.$$ 

Let $\theta_0, \theta_1$ be any two distinct real numbers and consider the LRT for testing $H_0 : \theta = \theta_0$
vs $H_1 : \theta = \theta_1$. Show that for some $\alpha^*(\theta_0, \theta_1)$, the rejection region for any $\alpha < \alpha^*$
is a bounded interval, and the rejection region for any $\alpha > \alpha^*$ is a union of two half
intervals. Find $\alpha^*$. Assume $\alpha \in (0, 1)$.

Hint: recall that $\frac{d}{dx} \arctan(x) = \frac{1}{1+x^2}$

(c) In the Cauchy location family, prove that, for any $\alpha < 0.5$ and $\theta_0 \in \mathbb{R}$, there exists no
UMP level-$\alpha$ test of $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.

(d) Optional: (not graded, no extra points) In words, can you explain why the optimal
LRT rejection regions take this odd form? Think about how you would explain to a
scientific collaborator why you are proposing such an odd test, beyond “it fell out of
an optimization problem.”

2. Some UMP(U) tests

Numerically find the UMP(U) test for the following hypothesis testing problems at level
$\alpha = 0.05$. For each problem,

(i) derive the appropriate test on paper,
(ii) numerically compute the cutoff value $c$ (and $\gamma$ if necessary), and
(iii) plot the power function of the level-$\alpha$ test for an appropriate range of parameter values.

(a) $X_i \overset{\text{ind}}{\sim} \text{Pois}(a_i \lambda)$ for $i = 1, \ldots, n$, where $a_1, \ldots, a_n$ are known positive constants and
$\lambda > 0$ is unknown. Test $H_0 : \lambda = 1$ vs. $H_1 : \lambda > 1$, with $n = 5$ and $a_i = i$. 

(b) \( X_i \sim N(\theta, \sigma_i^2) \) for \( i = 1, \ldots, n \), where \( \sigma_i^2 \) are known positive constants and \( \theta \in \mathbb{R} \) is unknown. Test \( H_0 : \theta = 0 \) vs. \( H_1 : \theta > 0 \), with \( n = 20 \) and \( \sigma_i^2 = i \). On your power plot, also plot the power function of the (sub-optimal) test that rejects for large \( \sum_i X_i \).

(c) \( X_1, \ldots, X_n \text{i.i.d.} \sim p_\theta(x) = \theta x^{-(1+\theta)}, \) for \( \theta > 0 \) and \( x > 1 \). Test \( H_0 : \theta = 1 \) vs. \( H_1 : \theta < 1 \), for \( n = 100 \). On your power plot, also plot the power function of the (sub-optimal) test that rejects for large \( \sum_i X_i \).

(d) \( X_1, \ldots, X_n \text{i.i.d.} \sim \text{Exp}(\theta) = \frac{1}{\theta} e^{-x/\theta}, \) for \( x, \theta > 0 \). Test \( H_0 : \theta = 1 \) vs. \( H_1 : \theta \neq 1 \), for \( n = 5 \). On your power plot, also plot the power function of the equal-tailed level-\( \alpha \) test. Finally, derive analytically the UMAU confidence interval for \( \theta \) and give the minimizing value of \( \tau \).

3. **Uniform UMP test**

Most UMP tests are for one-sided problems, but this problem gives an amusing example of a rare two-sided UMP test. Let \( X_1, \ldots, X_n \) be an i.i.d. sample from the uniform distribution on \( [0, \theta] \) for \( \theta > 0 \).

(a) Consider the problem of testing \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta > \theta_0 \). Show that any test \( \phi \) for which \( \phi(x) = 1 \) when \( x_{(n)} = \max\{x_1, \ldots, x_n\} > \theta_0 \) is UMP at level \( \alpha = E_{\theta_0}[\phi(X)] \).

(b) Now consider the problem of testing \( H_0 : \theta = \theta_0 \) against \( H_1 : \theta \neq \theta_0 \). Show that a unique UMP level-\( \alpha \) test exists, and is given by
\[
\phi(x) = 1\{x_{(n)} > \theta_0 \text{ or } x_{(n)} < \theta_0 \theta_0^{1/n}\}
\]

4. **Bayesian hypothesis testing**

\( p \)-values are commonly misinterpreted as representing “the probability that the null hypothesis is true, given the data.” This is an Bayesian statement and it depends on our prior beliefs. In fact, as this problem shows, even in a Bayesian setting, the \( p \)-value is generally not a good approximation for the posterior probability that the null is true.

Consider a univariate Gaussian problem with \( X \mid \theta \sim N(\theta, 1) \), where \( \theta = 0 \) under the null hypothesis and \( \theta \sim \Lambda_1 \) under the alternative hypothesis (assume \( \Lambda_1(\{0\}) = 0 \)). In addition let \( \pi_0 \) denote the a priori probability that the null hypothesis is true; therefore the full prior is a mixture between a point mass at 0 and \( \Lambda_1 \).

(a) Compute the posterior probability that the null hypothesis is true, i.e.
\[
\pi_{\text{post}}(x; \Lambda_1, \pi_0) = P(\theta = 0 \mid X = x).
\]

(b) Assume \( \pi_0 = 0.5 \) (we are initially agnostic between the null and the alternative), and find
\[
\pi^*_{\text{post}}(x) = \min_{\Lambda_1} \pi_{\text{post}}(x; \Lambda_1, 0.5),
\]
as a function of \( x \), for \( x > 0 \). Give the minimizing prior \( \Lambda_1 \), which also depends on \( x \).

**Note:** Obviously, waiting to see the data and then choosing the prior to maximize \( \pi_{\text{post}} \) is not a reasonable procedure for any analyst to carry out. The point is that \( \pi^*_{\text{post}} \) is a lower bound for anyone’s conditional belief that the null is true, who was not initially “biased” against the null.

(c) Now restrict \( \Lambda_1 = N(0, \tau^2) \) for \( \tau > 0 \), a class of more “realistic” priors. Compute \( \pi_{\text{post}} \) as a function of \( \tau^2 \) and \( x \). Find
\[
\pi^*_{\text{post},N}(x) = \min_{\tau^2 > 0} \pi_{\text{post}}(x; N(0, \tau^2), 0.5),
\]
and give the minimizing value of \( \tau^2 \), both as functions of \( x \), for \( x > 1 \).
(d) Now assume we observe a value of $X$ such that the two-sided $p$-value $p(X)$ (i.e., $p(x) = \Pr_0(|X| > |x|)$) takes the values 0.05, 0.01, 0.005, or 0.001. Numerically compute $\pi_{\text{post}}^*$ and $\pi_{\text{post}, N}^*$ for each value and make a small table. In words, interpret the results.

5. $p$-value densities
   Suppose $\mathcal{P}$ is a family with monotone likelihood ratio in $T(X)$, and the distribution of $T(X)$ is absolutely continuous with common support for all $\theta$. Let $\phi_\alpha$ denote the UMP level-$\alpha$ test of $H_0 : \theta \leq \theta_0$ vs. $H_0 : \theta > \theta_0$ that rejects when $T(X)$ is large. Let $p(X)$ denote the resulting $p$-value. Show that $p(X) \sim \text{Unif}[0, 1]$ if $\theta = \theta_0$, has non-increasing density on $[0, 1]$ if $\theta > \theta_0$, and has non-decreasing density on $[0, 1]$ if $\theta < \theta_0$. (If you are measure-theoretically troubled by the statement about densities, you can alternatively show that the CDF is linear, concave, or convex).