Stats 210A, Fall 2017
Homework 9

Due date: Thursday, Nov. 2

You may disregard measure-theoretic niceties about conditioning on measure-zero sets, almost-sure equality vs. actual equality, “all functions” vs. “all measurable functions,” etc. (unless the problem is explicitly asking about such issues).

Unless otherwise stated, assume asymptotic limits are taken as $n \to \infty$.

If I ask for a “limiting distribution,” I mean do an appropriate centering and scaling to find a limiting distribution that is non-degenerate (not converging in probability to a constant).

1. Some limiting distributions

In each case find a non-degenerate (non-deterministic) limiting distribution of $X_n$, appropriately normalized, as $n \to \infty$.

(a) $X_n \sim \text{Binom}(n, \theta)$.  
(b) $X_n \sim \text{Pois}(n\theta)$.  
(c) $X_n \sim \chi^2_n$. (You may use the mean and variance of a Gamma distribution.)  
(d) $X_n \sim \text{Binom}(n, \theta/n)$. (Hint: this one is Poisson and requires a different method than the others.)

2. Super-Efficient Estimator

Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, 1)$ and consider estimating $\theta$ via:

$$
\delta_n = \begin{cases} 
\bar{X}_n & |\bar{X}_n| > \frac{1}{4}n^{-1/4} \\
0 & |\bar{X}_n| \leq \frac{1}{4}n^{-1/4},
\end{cases}
$$

where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

Show that $\delta_n$ has the same asymptotic distribution as $\bar{X}_n$ when $\theta \neq 0$, but that $\sqrt{n}(\delta_n - 0) \overset{P}{\to} 0$ if $\theta = 0$.

(Not required, just for fun: is it possible to find a scaling for which $\delta_n$ converges to a non-degenerate distribution; i.e. not converging in probability to a constant?)

3. Wald Interval

For some model $P = \{P_\theta : \theta \in \Theta \subseteq \mathbb{R}\}$, with $\Theta$ open, suppose that we have an asymptotically normal estimator $\hat{\theta}$ with $\sqrt{n}(\hat{\theta} - \theta) \Rightarrow N(0, \sigma^2(\theta))$ (so the asymptotic variance of $\hat{\theta}$ depends on $\theta$).

Consider the confidence interval:

$$
C(X) = \left[ \hat{\theta} - \frac{\sigma(\hat{\theta})}{\sqrt{n}} z_{\alpha/2}, \quad \hat{\theta} + \frac{\sigma(\hat{\theta})}{\sqrt{n}} z_{\alpha/2} \right],
$$

where $z_{\alpha/2}$ is the upper $\alpha/2$ quantile of a standard normal distribution.

Prove that, if $\sigma^2(\theta)$ is continuous in $\theta$ and strictly positive, then for all $\theta$,

$$
P_\theta(\theta \in C(X)) \to 1 - \alpha.
$$

(Note: if $\hat{\theta}$ is the MLE, this is called a Wald interval)
4. Limiting Distribution of U-Statistics

Suppose $X_1, \ldots, X_n$ are i.i.d. random variables. $U_n = U_n(X_1, \ldots, X_n)$ is called a rank-2 U-statistic if

$$U_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} h(X_i, X_j)$$

where $h$ is a symmetric function with respect to $X_1$ and $X_2$, i.e. $h(x_1, x_2) = h(x_2, x_1)$ for any $x_1, x_2 \in \mathbb{R}$.

In this problem, we denote $\theta = \mathbb{E}h(X_1, X_2)$ and assume that $\mathbb{E}h(X_1, X_2)^2 < \infty$. Note that $U_n$ is the nonparametric UMVU estimator of $\theta$.

Perhaps surprisingly, we can derive the asymptotic distribution of $U_n$ in a relatively small number of steps using a technique called Hajek projection where we approximate it by an additive function of the independent $X_i$ variables. We walk through the proof below.

(a) Letting $g(x) = \mathbb{E}h(x, X_2) - \theta$, show that $\mathbb{E}g(X_i) = 0$ and $\text{Var}(g(X_i)) < \infty$, for all $i$.

(Note: Please carefully read the definition of the function $g : \mathbb{R} \to \mathbb{R}$. In particular, $g(X_1)$ is emphatically not the same thing as $\mathbb{E}h(X_1, X_2) - \theta$; one is a random variable and the other is a deterministic number, namely 0.)

(b) Define $\hat{U}_n = \theta + \frac{2}{n} \sum_{i=1}^{n} g(X_i)$. Show that $\mathbb{E}[(U_n - \hat{U}_n)g(X_i)] = 0$ for each $i$. (Note: in other words, $\hat{U}_n$ is the projection of $U_n$ onto the vector space consisting of random variables that are additive in $X_i$.)

(c) Show that $\sqrt{n}(U_n - \hat{U}_n) \xrightarrow{p} 0$ as $n \to \infty$. (Hint: show that $U_n$ and $\hat{U}_n$ have the same asymptotic variance, and then apply part (b)).

(d) Conclude that $\sqrt{n}(U_n - \theta) \Rightarrow \mathcal{N}(0, 4\zeta_1)$, where $\zeta_1 = \text{Var}(g(X_1))$.

(e) Assume that $\mathbb{E}X_i^4 < \infty$. Express the sample variance $S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ as a rank-2 U-statistic and use the above results to derive its asymptotic distribution.

(Note: a similar result holds in general for rank-$r$ U-statistics if we set $\hat{U} = \frac{r}{n} \sum_i g(X_i)$ where $g(x) = \mathbb{E}[h(x, X_2, \ldots, X_r)] - \theta$.)

5. Estimating an Inverse Mean

Suppose that $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, 1)$, and that we are interested in estimating the quantity $1/\theta$. In order to do so, we use the estimator $\delta(X) = 1/\bar{X}_n$ where $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ is the sample mean.

(a) Show that $\delta$ is asymptotically normal: specifically, that $\sqrt{n} \left(1/\bar{X}_n - 1/\theta\right) \Rightarrow \mathcal{N}(0, 1/\theta^4)$ for $\theta \neq 0$.

(b) Show that the expectation $\mathbb{E}|1/\bar{X}_n| = \infty$ for every $n$. Why does this not contradict the result of part (a)?