PROBLEM SESSION I

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The following session elaborates Lecture 2. The focus is on the relationship between Schur symmetric polynomials and uniform lozenge tilings of trapezoidal domains.

1. Preliminaries

In this section we gather some basic facts about Schur symmetric polynomials. We refer the reader to [1, Chapter 1] for a more detailed exposition, which goes well beyond what we will need.

A *signature* of length N is a sequence $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_N) \in \mathbb{Z}^N$ of integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$. We denote by $\mathbb{Y}_N$ the set of signatures of length N, which satisfy $\lambda_N \geq 0$ (these are sometimes referred to as *Young diagrams* or *partitions*). The *weight* of λ is given by $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_N$. There is a natural ordering on the space of signatures, called the *reverse lexicographic order*, given by

$$\lambda > \mu \iff \exists k \in \{1, \dots, N\} \text{ such that } \lambda_i = \mu_i, \text{ whenever } i < k \text{ and } \lambda_k > \mu_k.$$

Suppose that we have finitely many variables $x_1, \dots, x_N$. Let $x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ be a monomial and consider its antisymmetrization a_α: that is

$$a_\alpha = \sum_{\sigma \in S_N} \text{sign}(\sigma)\sigma(x^\alpha) = \sum_{\sigma \in S_n} \text{sign}(\sigma) x_{\sigma(1)}^{\alpha_1} \cdots x_{\sigma(N)}^{\alpha_N},$$

where S_N is the group of permutations of N elements and the action of $\sigma \in S_N$ on any polynomial is through permutation of the variables. Observe that a_α is skew-symmetric, i.e. $\sigma(a_\alpha) = \text{sign}(\sigma) \cdot a_\alpha$ and so $a_\alpha = 0$ unless $\alpha_1, \dots, \alpha_N$ are all distinct. We may thus assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ and therefore write $\alpha = \lambda + \delta$, where $\lambda \in \mathbb{Y}_N$ and $\delta = (N-1, N-2, \dots, 1, 0)$. We then have

$$a_\alpha = a_{\lambda+\delta} = \sum_\sigma \text{sign}(\sigma) \cdot \sigma(x^{\lambda+\delta}) = \det \left[x_i^{\lambda_j + N - j} \right]_{i,j=1}^N.$$

Since $a_{\lambda+\delta}$ is skew-symmetric it vanishes when $x_i = x_j$ for any $i \neq j$ and so $a_{\lambda+\delta}$ is divisible (over $\mathbb{Z}[x_1, \dots, x_N]$) by $x_i - x_j$ for $1 \leq i < j \leq N$ and consequently by their product, which is the *Vandermonde determinant*

$$\prod_{1 \leq i < j \leq N} (x_i - x_j) = \det \left[x_i^{N-j} \right]_{i,j=1}^N = a_\delta.$$

So $a_{\lambda+\delta}$ is divisible by a_δ in $\mathbb{Z}[x_1, \dots, x_N]$ and their quotient

$$s_\lambda(x_1, \dots, x_N) := \frac{a_{\lambda+\delta}}{a_\delta} = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{i,j=1}^N}{\prod_{1 \leq i < j \leq N} (x_i - x_j)} \tag{1}$$

is readily seen to be a *symmetric* and *homogeneous* polynomial of degree $|\lambda|$.

Definition (1) also makes sense for any $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$, so that the resulting $s_\lambda(x_1, \ldots, x_N)$ is a symmetric and homogeneous *Laurent* polynomial of degree $|\lambda| = \lambda_1 + \cdots + \lambda_N$. Indeed, we have the explicit relationship

$$s_{\lambda+r^N}(x_1, \ldots, x_N) = (x_1 \cdots x_N)^r \cdot s_\lambda(x_1, \ldots, x_N),$$

where $\lambda + r^N = (\lambda_1 + r, \cdots, \lambda_N + r)$ and $r \in \mathbb{Z}$.

We end this section with several of useful facts about Schur polynomials.

Fact 1. The set $\{s_\lambda(x_1, \ldots, x_N) : \lambda \in \mathbb{Y}_N\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}[x_1, \ldots, x_N]^{S_N} =: \Lambda_N$ (the space of symmetric polynomials with integer coefficients). Another popular basis for Λ_N is formed by the *monomial* symmetric polynomials

$$m_\lambda(x_1, \ldots, x_N) = \sum x^\alpha,$$

summed over all distinct permutations α of $\lambda = (\lambda_1, \ldots, \lambda_N)$, where again λ ranges over $\mathbb{Y}_N$.

Fact 2. The leading (in reverse lexicographic order) monomial in s_λ is m_λ.

Fact 3. If $\lambda = (k, 0, \ldots, 0) = (k)$ then $s_{(k)}(x_1, \ldots, x_N) = \sum_{|\lambda|=k} m_\lambda =: h_k$ – called the k-th *complete symmetric polynomial.*

Fact 4. If $\lambda = (\underbrace{1, \ldots, 1}_{\text{k}}, \underbrace{0, \ldots, 0}_{\text{N-k}}) = (1^k)$ then $s_{(1^k)}(x_1, \ldots, x_N) = \sum_{i_1 < \cdots < i_k} x_{i_1} \cdots x_{i_k} = m_{(1^k)} =: e_k$ – called the k-th *elementary symmetric polynomial.*

2. Problems

Problem 1. Prove the Weyl dimension formula

$$s_\lambda(1^N) := s_\lambda(\underbrace{1, \ldots, 1}_{\text{N}}) = \prod_{1 \leq i < j \leq N} \frac{\lambda_i - \lambda_j + j - i}{j - i} \tag{2}$$

by first computing $s_\lambda(1, q, q^2, \ldots, q^{N-1})$ and then letting $q \to 1$.

Problem 2. Prove the following branching rule for Schur symmetric polynomials

$$s_\lambda(x_1, \ldots, x_N) = \sum_{\mu \preceq \lambda} s_\mu(x_1, \ldots, x_{N-1}) x_N^{|\lambda|-|\mu|}, \tag{3}$$

where the sum is over $\mu = (\mu_1, \ldots, \mu_{N-1})$ such that $\mu_1 \leq \lambda_1 \leq \mu_2 \leq \lambda_2 \cdots \leq \mu_{N-1} \leq \lambda_N$ – such signatures are said to *interlace*.
Hint: Compute $s_\lambda(x_1, \ldots, x_{N-1}, 1)$ directly from (1) and apply properties of determinants to match the right side of (3); then use Fact 1 to finish.

Problem 3. We introduce a coordinate system on the hexagonal lattice with basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ as in Figure 1. Recall from lecture that a trapezoidal domain is one that is encoded by an N-tuple of integers $\ell = (\ell_1 > \ell_2 > \cdots > \ell_N) \in \mathbb{Z}^N$ – an example is given in Figure 1.
A tiling of the domain is a cover by lozenges of three types, given in the right part of Figure 2. Show that the total number of lozenge tilings is given by $s_\lambda(1^N)$, where $\ell_i = \lambda_i + N - i$.

Some ideas: Let the centers of the blue lozenges have coordinates $(N, \ell_1 + 1/2), \ldots, (N, \ell_N + 1/2)$. Suppose you place a particle in the center of each lozenge of Type 1 – so their coordinates in our basis are $(a, b+1/2)$ with $a, b \in \mathbb{Z}$. Try to argue that there are exactly k particles with first coordinate k in each tiling. If $(k, y_1^k + 1/2) \ldots, (k, y_k^k + 1/2)$ are the coordinates of particles on k-th column set $\lambda_i^k = y_i^k - k + i$ and convince yourself that $\lambda^k \preceq \lambda^{k+1}$ (notation from Problem 2). The latter gives a bijection between tilings and interlacing sequences of signatures – now use Problem 2.

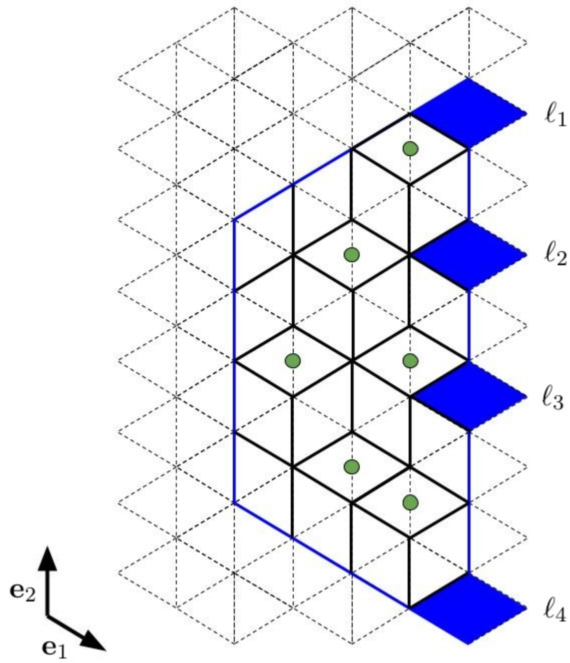

FIGURE 1. Example of a trapezoidal domain.

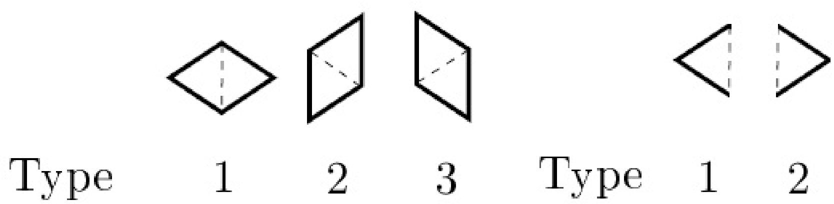

FIGURE 2. Types of lozenges and triangles.

Problem 4. Prove the famous Macmahon formula (see https://en.wikipedia.org/wiki/Plane_partition) that gives the number of tilings of a hexagon of sides A, B, C. Explicitly, show that it equals

$$\prod_{a=1}^{A}\prod_{b=1}^{B}\prod_{c=1}^{C}\frac{a+b+c-1}{a+b+c-2}.$$

Hint: Look at Figure 3.

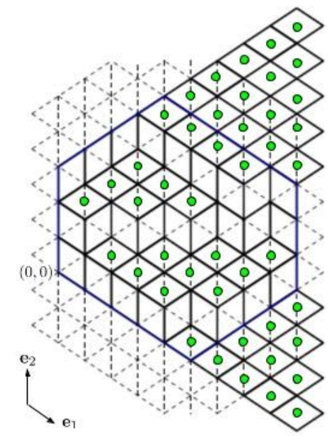

FIGURE 3. Tiling of hexagon.

Problem 5. Take $N \le \min(B, C)$ and consider the vertical section of the hexagon by the N-th vertical line from the left. There are precisely N horizontal lozenges on this vertical section. The positions of these lozenges in a uniformly random tiling form a random N-tuple of ordered integers $\ell_1 > \ell_2 > \cdots > \ell_N$. Prove that the probability distribution on such N-tuples has the form

$$\mathbb{P}(\ell_1, \ldots, \ell_N) = \frac{1}{Z}\prod_{1\le i<j\le N}(\ell_i-\ell_j)^2\prod_{i=1}^{N} w(\ell_i), \tag{4}$$

find explicit expressions for Z and w.

Hint: When one fixes the horizontal lozenges along a vertical line, the tiling splits into two: to the left and to the right of this line. The left and right tilings then can be counted by the same trick as in Problem 3.

REFERENCES

[1] Macdonald, I.G. "Symmetric functions and Hall polynomials". Second Edition. Oxford University Press, 1999.