Tilings, matrices, and representations through Schur generating functions.

Vadim Gorin
MIT (Cambridge) and IITP (Moscow)

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We study random **ordered** N–tuples of reals or integers

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Key tool: **Schur** generating functions and generalizations
A new version of the Fourier transform

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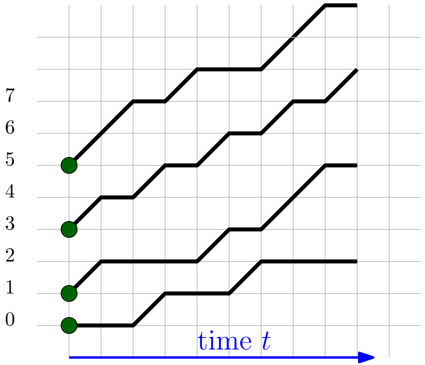
TODAY: Examples of stochastic systems and results.
Next lectures: Detailed math.

Example 1: Noncolliding random walks



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- probability of jump p
- started at *arbitrary* lattice points
- conditioned **never to collide**

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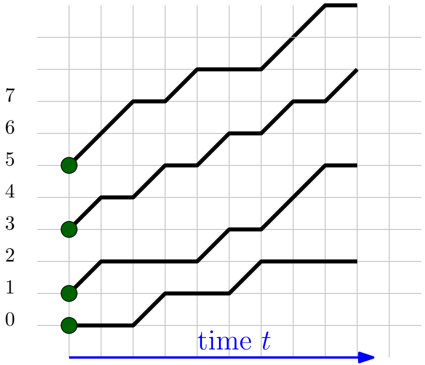

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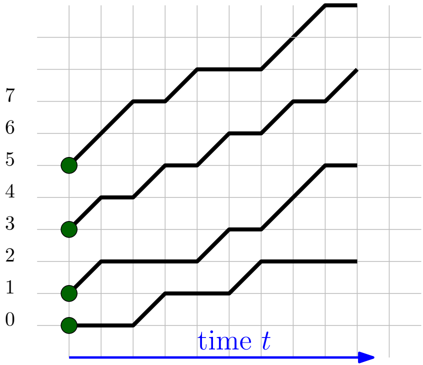

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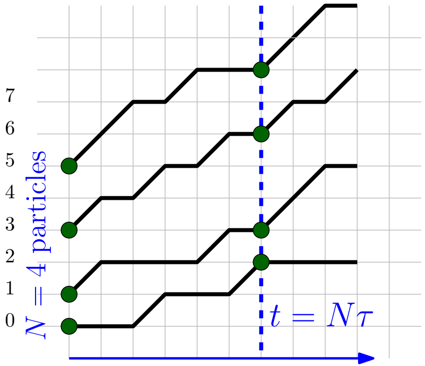
Solution: Consider $\lim_{T\to\infty}$ ("No collisions up to time T")

Example 1: Noncolliding random walks

Macroscopic behavior of paths at time $t = \tau N$ as $N \to \infty$?



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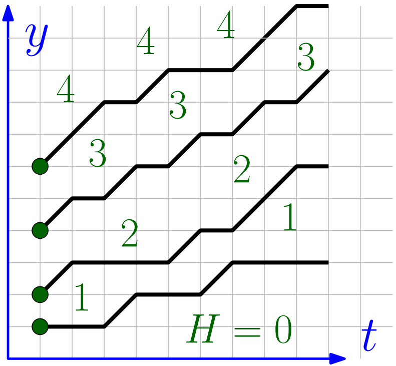

Theorem. (Bufetov–G.–13-17)
Suppose that the **height function** satisfies at $t = 0$:

$$\lim_{N \to \infty} \frac{1}{N} H(0, y \cdot N) \to \mathfrak{h}(0, y)$$

(For all $y > 0$.)

Then

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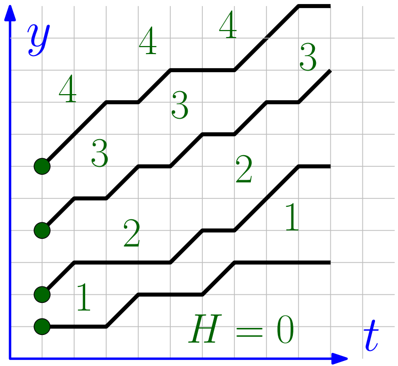

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Then for **deterministic** $\mathfrak{h}(\tau, y)$, generalized Gaussian field $\xi(\tau, y)$

Law of Large Numbers: $\lim_{N\to\infty} \frac{1}{N} H(\tau \cdot N, y \cdot N) = \mathfrak{h}(\tau, y)$

CLT: $\lim_{N\to\infty} \left[H(\tau \cdot N, y \cdot N) - \mathbb{E} H(\tau \cdot N, y \cdot N) \right] = \xi(\tau, y)$

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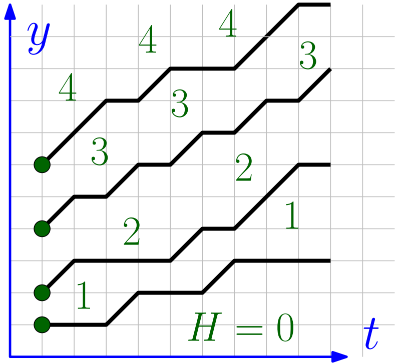

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Important: No rescaling in CLT!

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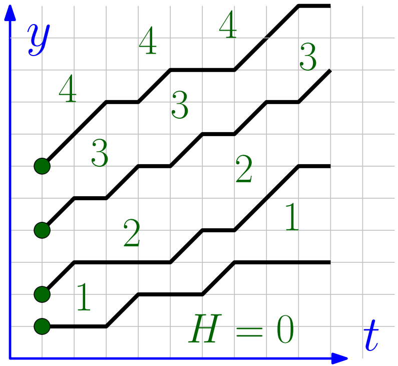

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Answers $\mathfrak{h}(\tau, y)$, $\xi(\tau, y)$: explicit non-trivial dependence on $\mathfrak{h}(0, y)$.

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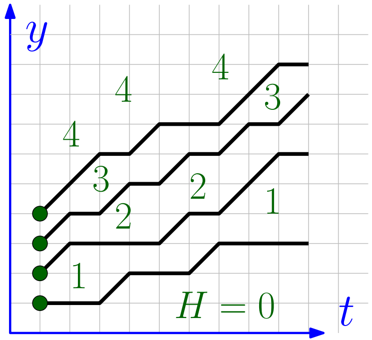


Theorem. (Borodin-Ferrari-08) Suppose that the **height function** satisfies at $t = 0$:

$$\lim_{N\to\infty} \frac{1}{N} H(0, y\cdot N) \to y, \quad 0 < y < 1$$

[Densely packed initial condition]

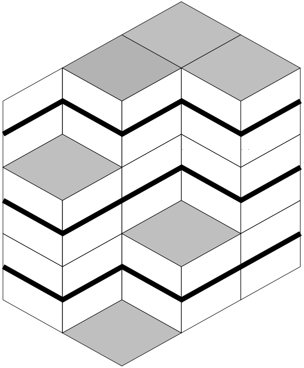
The fluctuation field $\xi(\tau, y)$:

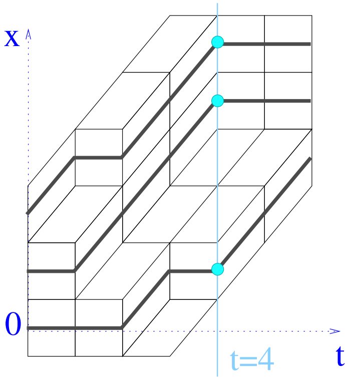
- Vanishes outside the domain $(1-\sqrt{\tau})^2 < y < (1+\sqrt{\tau})^2$.
- Inside the domain is identified with the pullback of the $2d$ Gaussian Free Field with Dirichlet boundary conditions in the upper half-plane $\mathbb{H}$ with respect to an explicit map Ω.

Covariance for **GFF** in $\mathbb{H}$: $$\mathbb{E}G(z)G(w) = -\frac{1}{2\pi}\ln\left|\frac{z-w}{z-\bar{w}}\right|.$$

Example 2: Lozenge tilings

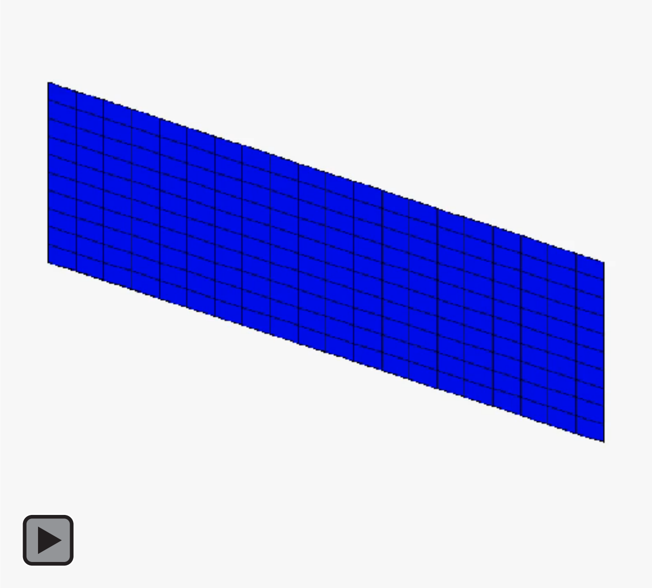
Let paths start **and** end densely packed.





These are **uniformly random lozenge tilings** of a hexagon.

Example 2: Lozenge tilings



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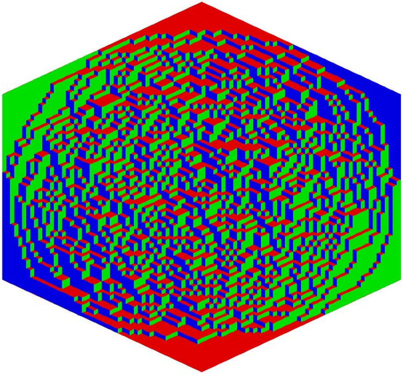

Theorem. (Cohn-Larsen-Propp-98)
The height function of uniformly random lozenge tilings of a hexagon converges to an explicit **deterministic** limit shape as the mesh size goes to 0.

Frozen outside inscribed circle.

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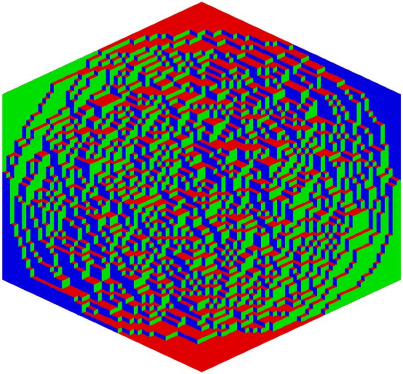


Theorem. (Cohn-Larsen-Propp-98) The height function of uniformly random lozenge tilings of a hexagon converges to an explicit **deterministic** limit shape as the mesh size goes to 0.

Frozen outside inscribed circle.

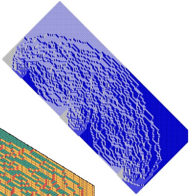
Theorem. Centered height function converges in non-frozen region to a Gaussian Field — pullback of **GFF** with an explicit map Ω.

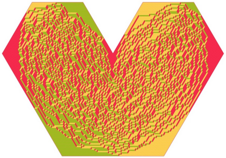
(Kenyon–Okounkov conjectured; Petrov-12, Duits-15, Bufetov-Gorin-16 proved)

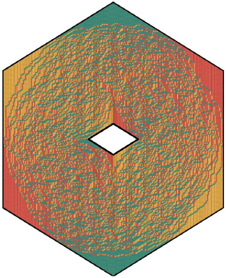
Example 2: Random tilings of general domains

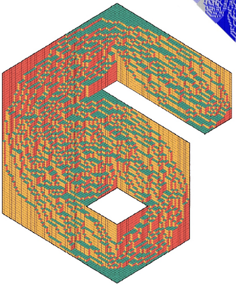
Theorem. Deterministic limit shape + algorithmic description

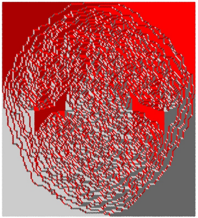
(Cohn-Kenyon-Propp-01), (Kenyon-Okounkov-05)











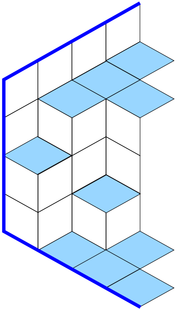
Thanks to Alisa Knizel, Sevak Mkrtchyan, Leonid Petrov

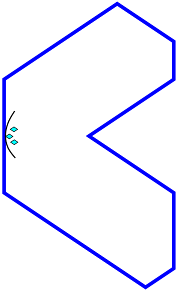
Conjecture-Theorem. The **Gaussian Free Field** fluctuations.

(Kenyon-Okounkov-05), (Borodin-Ferrari-08), (Petrov-12), (Bufetov-Gorin-16,17), (Bufetov-Knizel-16)

Example 2: From lozenge tilings to GUE

Local features of uniformly random tilings?





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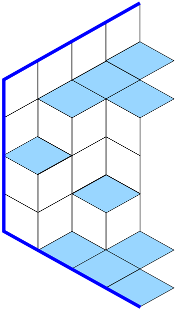

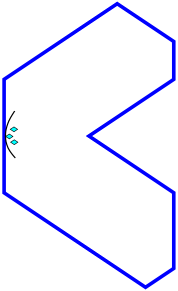

Local features of uniformly random tilings?

Conjecture–Theorem. As the mesh $\varepsilon \to 0$, after $\sqrt{\varepsilon}$ rescaling, the interlacing particles near boundary converge to **GUE–corners process**.

(Johnasson-Nordenstam-06), (Okounkov-Reshetikhin-06),

(Gorin-Panova-13), (Novak-14)

X — matrix of i.i.d. standard **complex** Gaussians. **Hermitian** matrix $A = (X + X^*)/2$:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

GUE–corners process =

eigenvalues of principal corners.

Example 3: Random matrices

- $A = (X + X^*)/2$, with X — $N \times N$ matrix of i.i.d. standard complex Gaussians $N(0, t) + \mathbf{i}N(0, t)$.
- B — deterministic $N \times N$ with eigenvalues $b_1, \ldots, b_N$.

What can we do with two square matrices?

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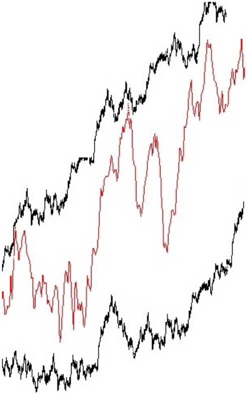

$$C = A + B$$

Lemma. The eigenvalues of C are **Dyson Brownian Motion** particles at time t, when started from $(b_1, \dots, b_N)$ at time 0.

DBM = N independent Brownian Motions conditioned on no collisions.

Continuous noncolliding random walks

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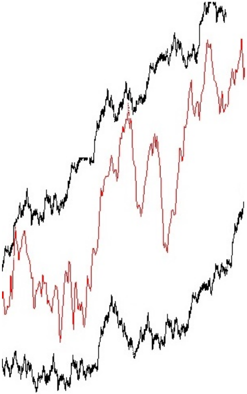

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Theorem. Eigenvalues of C satisfy macroscopic **LLN** and **CLT**.

Example 3: Random matrices

$$A = \begin{pmatrix} a_1 & 0 & & \\ 0 & a_2 & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & a_N \end{pmatrix} \qquad B = \begin{pmatrix} b_1 & 0 & & \\ 0 & b_2 & 0 & \\ & 0 & \ddots & 0 \\ & & 0 & b_N \end{pmatrix}$$

$U, V -$ Haar–random in $Unitary(N; \mathbb{R} / \mathbb{C} / \mathbb{H})$

$$\boxed{C = UAU^* + VBV^*}$$

$\nwarrow \quad \nearrow$

uniformly random eigenvectors

Question: What can you say about **eigenvalues** of C?

Example 3: Random matrices as $N \to \infty$

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$$\lim_{N\to\infty} \boxed{C = UAU^* + VBV^*}$$

Theorem. (Voiculescu, 80s) The **empirical measure** (=derivative of height function) of eigenvalues of C is deterministic as $N \to \infty$.

$$\mu_A = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i} \quad \mu_B = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{b_i} \quad \mu_C = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{c_i}$$

$$G_\mu(z) = \int \frac{\mu(dx)}{z-x}, \qquad R_\mu(z) = \Big(G_\mu(z)\Big)^{-1} - \frac{1}{z}.$$

$$R_{\mu_C}(z) = R_{\mu_A}(z) + R_{\mu_B}(z).$$

Free convolution via **Voiculescu R–transform**.

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Many ways to continue after Voiculescu's LLN:

- The Gaussian **fluctuations** for $\boxed{C = UAU^* + VBV^*}$ — second order freeness of (Collins-Mingo-Sniady-Speicher-06).

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$$\lambda_1 > \lambda_2 > \cdots > \lambda_N, \quad \lambda_i \in \mathbb{Z}. \qquad T_\lambda \otimes T_\nu = \bigoplus_\kappa c^\kappa_{\lambda,\nu} T_\kappa$$

Littlewood–Richardson coefficients $c^\kappa_{\lambda,\nu}$ hard as $N \to \infty$.

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Approach of (Biane-95), (Bufetov-Gorin-13), (Collins-Novak-Sniady-16):

Random κ through $$P(\kappa) = \frac{\dim(T_\kappa) c^\kappa_{\lambda,\nu}}{\dim T_\lambda \cdot \dim T_\nu}.$$

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Semi-classical limit degenerates representations of a Lie group into orbital measures on its Lie algebra

$$T_\lambda \otimes T_\nu \longrightarrow UAU^* + VBV^*$$

Example 4: Random matrices as $\beta(=1,2,4) \to \infty$

Theorem. (Gorin–Marcus–17) Eigenvalues of C **crystallize** (= become deterministic) as dimension of the base field $\beta \to \infty$:

$$\lim_{\beta\to\infty} \boxed{C = UAU^* + VBV^*}$$

$$\prod_{i=1}^{N}(z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N}(z - a_i - b_{\sigma(i)})$$

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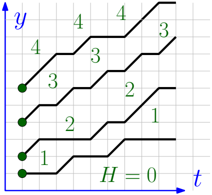
How to add matrices over β**–dimensional field**????

Summary of examples



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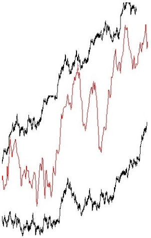

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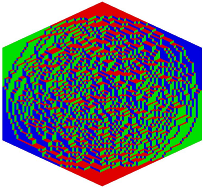
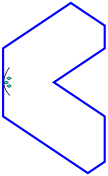


$$T_\lambda \otimes T_\nu = \bigoplus_\kappa c^\kappa_{\lambda,\nu} T_\kappa$$



Recurring question: **asymptotic** behavior of **random** N–tuple

$$\lambda_1 > \lambda_2 > \cdots > \lambda_N$$

Our aim: uniform approach to the analysis.

Reminder: characteristic functions, Fourier transform

A classical powerful tool:

Random variable $\xi \longleftrightarrow \phi_\xi(t) = \mathbb{E}\exp(\mathbf{i}t\xi)$

- The law of ξ is uniquely determined by ϕ_ξ.
- Works nicely with addition of **independent** ξ_1, ξ_2:

$$\phi_{\xi_1+\xi_2}(t) = \phi_{\xi_1}(t) \cdot \phi_{\xi_2}(t)$$

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Application: proof of the CLT for i.i.d. random variables ξ_i

$$S[N] = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} (\xi_i - \mathbb{E}\xi_i)$$

$$\phi_{S[N]}(t) = \left(1 - \mathrm{Var}(\xi_i)\frac{t^2}{2N} + O\left(N^{-3/2}\right)\right)^N \to \exp\left(-\mathrm{Var}(\xi_i)\frac{t^2}{2}\right).$$

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Can not work this way in our framework, but **there is an analogue!**

Schur generating functions

$\mathbb{P}(\cdot)$ — probability measure on

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \in \mathbb{Z}^N.$$

Its **Schur generating function** is

$$\mathcal{G}_{\mathbb{P}} = \sum_{\lambda} \mathbb{P}(\lambda) \frac{s_\lambda(x_1, \ldots, x_N)}{s_\lambda(1, \ldots, 1)}, \qquad s_\lambda(x_1, \ldots, x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}.$$

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$N = 1$ — moment generating function. $\quad \mathcal{G}_{\mathbb{P}} = \sum_{k} \mathbb{P}(k) x^k$

$x = \exp(\mathbf{i}t)$ turns it into the conventional **characteristic function**.

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How is it good?

1. The distribution $\mathbb{P}$ can be **efficiently** reconstructed from $\mathcal{G}_{\mathbb{P}}$.
   - Exactly at finite N.
   - Asymptotically as $N \to \infty$.
2. $\mathcal{G}_{\mathbb{P}}$ changes nicely upon operations on the system, such as:
   - Evolution of non-colliding random walks. ($*$ by a function)
   - Moving section in a random tiling. (plug in some $x_i = 1$)
   - Computing tensor products, adding/multiplying matrices. ($*$)

Schur generating functions: Uniqueness

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Lemma. Fixed N. $\mathcal{G}_{\mathbb{P}}$ on the torus $|x_i| = 1$ uniquely determines P.

Proof. Scalar product — integral against uniform measure on $\mathbb{T}^N$.

$$\langle f, g \rangle = \iiint_{|x_i|=1} f(x_1, \ldots, x_N) \cdot \overline{g(x_1, \ldots, x_N)} \prod_{i<j} |x_i - x_j|^2$$

Then $\langle s_\lambda, s_\mu \rangle = N! \cdot \delta_{\lambda,\mu}$, hence

$$\mathbb{P}(\lambda) = \frac{1}{N!} \langle s_\lambda, \mathcal{G}_P \rangle. \quad \square$$

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Conceptual: Reminiscent of direct/inverse Fourier transform.
This is **harmonic analysis** on the unitary group $U(N)$.

Schur generating functions: $N \to \infty$

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General idea: Asymptotic probabilistic characteristics of λ's are in correspondence with finite–dimensional features of $\mathcal{G}_{\mathbb{P}}$ as $N \to \infty$.

There are **topological choices** to be made depending on the desired asymptotic regime.

Asymptotic statement of (Bufetov–Gorin–13,16,17)

$$\mathcal{G} = \sum_{\ell} \mathbb{P}(\ell) \frac{s_\ell(x_1, \dots, x_N)}{s_\ell(1, \dots, 1)}$$

- $\frac{1}{N} (\partial_i)^a \ln(\mathcal{G})\Big|_{x_1=\dots=x_N=1} \to c_a$
- $(\partial_i)^a (\partial_j)^b \ln(\mathcal{G})\Big|_{\dots=1} \to d_{a,b}$
- $[\prod_{a=1}^k \partial_{i_a}] \ln(\mathcal{G})\Big|_{=1} \to 0, \ |\{i_a\}| > 2$

if and only if

$$p_k = \sum_{i=1}^{N} \left(\frac{\ell_i}{N}\right)^k$$

- $\frac{1}{N} p_k \to \mathfrak{p}(k)$
- $\mathbb{E} p_k p_m - \mathbb{E} p_k \mathbb{E} p_m \to \mathfrak{cov}(k, m)$
- $p_k - \mathbb{E} p_k \to$ Gaussians

$$\mathfrak{p}(k) = [z^{-1}] \frac{1}{(k+1)(1+z)} \left(\frac{1+z}{z} + (1+z) \sum_{a=1}^{\infty} \frac{c_a z^{a-1}}{(a-1)!} \right)^{k+1}$$

$$\mathfrak{cov}(k, m) = [z^{-1} w^{-1}] \left(\left(\sum_{a=0}^{\infty} \frac{z^a}{w^{1+a}} \right)^2 + \sum_{a,b=1}^{\infty} \frac{d_{a,b}}{(a-1)!(b-1)!} z^{a-1} w^{b-1} \right)$$

$$\times \left(\frac{1+z}{z} + (1+z) \sum_{a=1}^{\infty} \frac{c_a z^{a-1}}{(a-1)!} \right)^k \left(\frac{1+w}{w} + (1+w) \sum_{a=1}^{\infty} \frac{c_a w^{a-1}}{(a-1)!} \right)^m$$

Coming next:

Schur generating functions (=harmonic analysis on $U(N)$) as a tool in $2d$ statistical mechanics and random matrix theory.

We will develop theory in three examples:

- **Lecture 2**: Gaussian Unitary Ensemble as a limit in uniformly random tilings.
- **Lecture 3**: Addition of large independent random matrices leading to the free convolution.
- **Lecture 4**: $\beta \to \infty$ limit of random matrix operations leading to polynomial operations preserving real–rootedness ("finite free probability")