General beta random matrix theory
(at MATRIX Institute)

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Lecture 2
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Roadmap

• What are general $\beta$ random matrices?
• Lecture 1: corners of $\beta$ random matrices.
• Problem set 1.
• Lecture 2: sums of $\beta$ random matrices.
• Problem set 2.
• Lecture 3: questions and discussion of problem sets.

[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

This is NOT a research talk about brand new results.
Instead we explore basic structures and definitions.
(See “Lattice Paths, Combinatorics and Interactions” in 2 weeks).
Recap: $\beta$–corners process

Fix $\beta > 0$
$N = 1, 2, \ldots$
$a_1, \ldots, a_N \in \mathbb{R}$

**Definition.** Eigenvalues of corners of $N \times N$ random $\beta$-matrix with uniformly random eigenvectors and fixed eigenvalues $(a_i)_{i=1}^N$ are a triangular array $(x_i^k)_{1 \leq i \leq N}$ satisfying

$$x_{i+1}^k \leq x_i^k \leq x_{i+1}^{k+1}, \quad (x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N),$$

with distribution of density

$$\left[ \prod_{k=1}^N \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k} \right] \prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^k \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$
Toy question: sum of independent random variables
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**Theorem.** Random $N \times N$ self-adjoint independent matrices $A, B$. The law of the sum $C = A + B$ is uniquely determined by

$$
\mathbb{E} \exp (i \text{Trace}(CZ)) = \mathbb{E} \exp (i \text{Trace}(AZ)) \cdot \mathbb{E} \exp (i \text{Trace}(BZ)),
$$

which should be valid for each self-adjoint $Z$.

**Proof.**
Reduction to eigenvalues

Definition 1. $A$: deterministic eigenvalues $(a_1, \ldots, a_N)$ and uniformly random eigenvectors (invariant under $A \mapsto UAU^*$).

Then law of $\text{Trace}(AZ)$ depends only on eigenvalues $(z_i)_{i=1}^N$ of $Z$ and we define the **multivariate Bessel function** through

$$B_{a_1,\ldots,a_N}(iz_1, \ldots, iz_N; \beta/2) = \mathbb{E} \exp(i \text{Trace}(AZ))$$

Proof.
Reduction to corners

Fix \( \beta > 0 \)

\( N = 1, 2, \ldots \)

\( a_1, \ldots, a_N \in \mathbb{R} \)

**Definition 2.** Take \( \beta \)-corners process with top row \((a_i)_{i=1}^{N}\); \((x_i^k)_{1 \leq i \leq k \leq N}\). The **multivariate Bessel function** is:

\[
B_{a_1, \ldots, a_N}(z_1, \ldots, z_N; \beta/2) = \mathbb{E} \exp \left[ \sum_{k=1}^{N} z_k \left( \sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1} \right) \right]
\]

**Important:** This makes sense for each \( \beta > 0 \).
Proposition. Two definitions coincide, i.e., at $\beta = 1, 2, 4$ we have

$$\mathbb{E} \exp (i \text{Trace}(AZ)) = \mathbb{E} \exp \left[ i \sum_{k=1}^{N} z_k \left( \sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1} \right) \right]$$

Proof.
Eigenvalues of the sum of $\beta$ random matrices

**Definition.** Given deterministic eigenvalues $(a_i)_{i=1}^N$ and $(b_i)_{i=1}^N$ we define (random) eigenvalues $(c_i)_{i=1}^N$ of the sum of independent $\beta$-matrices with uniformly random eigenvectors through

$$
\mathbb{E} B_{c_1,\ldots,c_N}(z_1,\ldots,z_N; \beta/2) = B_{a_1,\ldots,a_N}(z_1,\ldots,z_N; \beta/2) \cdot B_{b_1,\ldots,b_N}(z_1,\ldots,z_N; \beta/2)
$$

- $c = a \boxplus_\beta b$ at $\beta = 1, 2, 4$ is the same old addition.
- At general $\beta > 0$ one needs to show the existence of **probability measure** defining $(c_i)_{i=1}^N$.
- It is well-defined as a generalized function (distribution), but being a measure is a known open problem.

[$\approx$ need positivity of structure constants of multiplication for Macdonald polynomials]
Example: $\beta$–addition at $N = 1$. 
What are $B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; \frac{\beta}{2})$?

- **Symmetric** functions in $z_1, \ldots, z_N$.
- Limits of **Jack or Macdonald polynomials**.
  
  $N = 1$:
  
  $$e^{az} = \lim_{m \to \infty} (1 + z/m)^{[ma]}$$

- Explicit Taylor **series expansion** in Jack polynomials.
  
  $N = 1$:
  
  $$e^{az} = 1 + az + \frac{(az)^2}{2!} + \frac{(az)^2}{3!} + \ldots$$

$$B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; \frac{\beta}{2}) = \sum_{\mu} \frac{P_\mu(z_1, \ldots, z_N; \frac{\beta}{2}) Q_\mu(a_1, \ldots, a_N; \frac{\beta}{2})}{(N\frac{\beta}{2})_\mu}$$

- **Eigenfunctions** of (symmetric) Dunkl operators

  $$D_i := \frac{\partial}{\partial z_i} + \frac{\beta}{2} \sum_{j:j \neq i} \frac{1}{z_i - z_j} \circ (1 - s_{i,j})$$

$$\sum_{i=1}^{N} (D_i)^k B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; \frac{\beta}{2}) = \sum_{i=1}^{N} (a_i)^k B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; \frac{\beta}{2})$$
**Theorem:** At $\beta = 0$ the operation $(a, b) \mapsto c = a \oplus_0 b$ has the form: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

**Proof I.**
Theorem: At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ has the form:

Choose a permutation $\sigma \in S(N)$ uniformly at random and set

$$(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)}).$$

Proof II.
Expected characteristic polynomial

**Theorem.** At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ is:
Choose a permutation $\sigma \in S(N)$ uniformly at random and set
$(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

**Corollary.** At $\beta = 0$, we have
\[
\mathbb{E} \prod_{i=1}^{N} (z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N} (z - a_i - b_{\sigma(i)}).
\]

**Theorem.** The last expectation identity holds for all $\beta \in [0, +\infty]$.  
[At $\beta = \infty$, expectation sign can be removed.]
Expected characteristic polynomial

**Theorem.** At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ is: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

**Corollary.** At $\beta = 0$, we have

$$\mathbb{E} \prod_{i=1}^{N} (z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N} (z - a_i - b_{\sigma(i)}).$$

**Theorem.** The last expectation identity holds **for all** $\beta \in [0, +\infty]$.

[At $\beta = \infty$, expectation sign can be removed.]

**Hint on the proof.**

- Expectations of Jack polynomials in eigenvalues $(c_1, \ldots, c_N)$.
- One-column Jacks do not depend on $\beta$:
  $$P_{(1^k)}(c_1, \ldots, c_N; \frac{\beta}{2}) = e_k(c_1, \ldots, c_N).$$
- There are coefficients of expected characteristic polynomial.
Another asymptotic result: free convolution

**Theorem.** Suppose that as $N \to \infty$

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{a_i/N} \to \mu_a, \quad \text{with} \quad G_{\mu_a}(z) = \int \frac{\mu_a(dx)}{z - x},$$

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{b_i/N} \to \mu_b, \quad \text{with} \quad G_{\mu_b}(z) = \int \frac{\mu_b(dx)}{z - x}.$$

Then for $c = a \boxplus_{\beta} b$

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{c_i/N} \to \mu_c, \quad \text{with} \quad G_{\mu_c}(z) = \int \frac{\mu_c(dx)}{z - x},$$

$$R_{\mu}(z) = (G_{\mu}(z))^{-1} - \frac{1}{z}, \quad R_{\mu_c}(z) = R_{\mu_a}(z) + R_{\mu_b}(z).$$

Holds for each $\beta > 0$, but **not** for $\beta = 0$.

[Come back to my talk in two weeks for the critical $\beta N \to \gamma$ regime.]
End of Lecture 2.  Don’t forget about Problem set 2.
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