General beta random matrix theory

(at MATRIX Institute)

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Lecture 2
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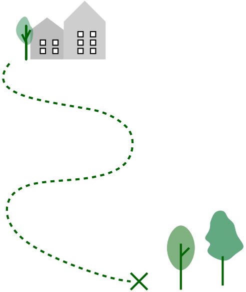
Roadmap



- What are general β random matrices?
- Lecture 1: corners of β random matrices.
- Problem set 1.
- Lecture 2: sums of β random matrices.
- Problem set 2.
- Lecture 3: questions and discussion of problem sets.

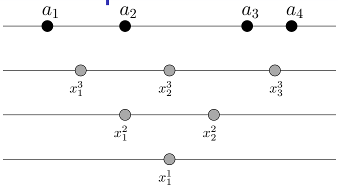
[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

This is NOT a research talk about brand new results.
Instead we explore **basic structures and definitions.**

(See "Lattice Paths, Combinatorics and Interactions" in 2 weeks).

Recap: β–corners process



Fix $\beta > 0$
$N = 1, 2, \dots$
$a_1, \dots, a_N \in \mathbb{R}$

Definition. Eigenvalues of corners of $N \times N$ **random β-matrix** with uniformly random eigenvectors and fixed eigenvalues $(a_i)_{i=1}^N$ are a triangular array $(x_i^k)_{1 \le i \le N}$ satisfying

$$x_{i+1}^k \le x_i^k \le x_{i+1}^{k+1}, \qquad (x_1^N, \dots, x_N^N) = (a_1, \dots, a_N),$$

with distribution of density

$$\left[\prod_{k=1}^{N} \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k}\right] \cdot \prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

Next question: What is the sum of random β-matrices?

Toy question: sum of independent random variables

Toy question: sum of independent random variables II

Sum of matrices at $\beta = 1, 2, 4.$

Theorem. Random $N \times N$ self-adjoint independent matrices A, B. The law of the sum $C = A + B$ is uniquely determined by

$$\boxed{\mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(CZ)\right) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right) \cdot \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(BZ)\right)},$$

which should be valid for each self-adjoint Z.

Proof.

Reduction to eigenvalues

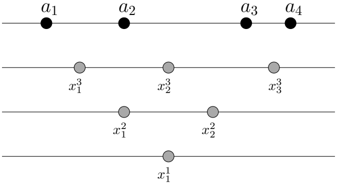
Definition 1. A: deterministic eigenvalues $(a_1, \ldots, a_N)$ and uniformly random eigenvectors (invariant under $A \mapsto UAU^*$). Then law of $\mathrm{Trace}(AZ)$ depends only on eigenvalues $(z_i)_{i=1}^N$ of Z and we define the **multivariate Bessel function** through

$$\boxed{B_{a_1,\ldots,a_N}(\mathbf{i}z_1, \ldots, \mathbf{i}z_N;\ \beta/2) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right)}$$

Proof.

Reduction to corners

Fix $\beta > 0$
$N = 1, 2, \ldots$
$a_1, \ldots, a_N \in \mathbb{R}$



Definition 2. Take β–corners process with top row $(a_i)_{i=1}^N$; $(x_i^k)_{1 \le i \le k \le N}$. The **multivariate Bessel function** is:

$$\boxed{B_{a_1,\ldots,a_N}(z_1, \ldots, z_N;\ \beta/2) = \mathbb{E} \exp \left[\sum_{k=1}^{N} z_k \left(\sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1} \right) \right]}$$

Important: This makes sense for each $\beta > 0$.

Proposition. Two definitions coincide, i.e., at $\beta = 1, 2, 4$ we have

$$\mathbb{E} \exp \left(\mathbf{i} \mathrm{Trace}(AZ) \right) = \mathbb{E} \exp \left[\mathbf{i} \sum_{k=1}^{N} z_k \left(\sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1} \right) \right]$$

Proof.

Eigenvalues of the sum of β random matrices

Definition. Given deterministic eigenvalues $(a_i)_{i=1}^N$ and $(b_i)_{i=1}^N$ we define (random) eigenvalues $(c_i)_{i=1}^N$ of the sum of independent β-matrices with uniformly random eigenvectors through

$$\mathbb{E} B_{c_1,\dots,c_N}(z_1,\dots,z_N;\,\beta/2) \\ = B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,\beta/2)\cdot B_{b_1,\dots,b_N}(z_1,\dots,z_N;\,\beta/2)$$

- $c = a \boxplus_\beta b$ at $\beta = 1, 2, 4$ is the same old addition.
- At general $\beta > 0$ one needs to show the existence of **probability measure** defining $(c_i)_{i=1}^N$.
- It is well-defined as a generalized function (distribution), but being a measure is a known open problem.

[$\approx$ need positivity of structure constants of multiplication for **Macdonald polynomials**]

Example: β–addition at $N = 1$.

What are $B_{a_1,\dots,a_N}(z_1,\dots,z_N;\ \frac{\beta}{2})$?

- **Symmetric** functions in $z_1,\dots,z_N$.
- Limits of **Jack or Macdonald polynomials**.

  $N=1:$ $$e^{az} = \lim_{m\to\infty}(1+z/m)^{\lfloor ma\rfloor}$$
- Explicit Taylor **series expansion** in Jack polynomials.

  $N=1:$ $$e^{az} = 1 + az + \frac{(az)^2}{2!} + \frac{(az)^2}{3!} + \dots$$

$$B_{a_1,\dots,a_N}(z_1,\dots,z_N;\ \tfrac{\beta}{2}) = \sum_{\mu} \frac{P_\mu(z_1,\dots,z_N;\ \frac{\beta}{2})Q_\mu(a_1,\dots,a_N;\ \frac{\beta}{2})}{(N\frac{\beta}{2})_\mu}$$

- **Eigenfunctions** of (symmetric) Dunkl operators

$$D_i := \frac{\partial}{\partial z_i} + \frac{\beta}{2}\sum_{j:j\neq i}\frac{1}{z_i - z_j}\circ(1 - s_{i,j})$$

$$\sum_{i=1}^{N}(D_i)^k B_{a_1,\dots,a_N}(z_1,\dots,z_N;\ \tfrac{\beta}{2}) = \sum_{i=1}^{N}(a_i)^k B_{a_1,\dots,a_N}(z_1,\dots,z_N;\ \tfrac{\beta}{2})$$

Theorem: At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ has the form: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

Proof I.

Theorem: At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ has the form: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \dots, c_N) = (a_1 + b_{\sigma(1)}, \dots, a_N + b_{\sigma(N)})$.

Proof II.

Expected characteristic polynomial

Theorem. At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ is: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \dots, c_N) = (a_1 + b_{\sigma(1)}, \dots, a_N + b_{\sigma(N)})$.

Corollary. At $\beta = 0$, we have

$$\mathbb{E} \prod_{i=1}^{N} (z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N} (z - a_i - b_{\sigma(i)}).$$

Theorem. The last expectation identity holds **for all $\beta \in [0, +\infty]$.**

[At $\beta = \infty$, expectation sign can be removed.]

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Hint on the proof.

- Expectations of Jack polynomials in eigenvalues $(c_1, \dots, c_N)$.
- One-column Jacks do not depend on β: $P_{(1^k)}(c_1, \dots, c_N; \frac{\beta}{2}) = e_k(c_1, \dots, c_N)$.
- There are coefficients of expected characteristic polynomial.

Another asymptotic result: free convolution

Theorem. Suppose that as $N \to \infty$

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{a_i/N} \to \mu_a, \qquad \text{with} \qquad G_{\mu_a}(z) = \int \frac{\mu_a(dx)}{z-x},$$

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{b_i/N} \to \mu_b, \qquad \text{with} \qquad G_{\mu_b}(z) = \int \frac{\mu_b(dx)}{z-x}.$$

Then for $c = a \boxplus_\beta b$

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{c_i/N} \to \mu_c, \qquad \text{with} \qquad G_{\mu_c}(z) = \int \frac{\mu_c(dx)}{z-x},$$

$$R_\mu(z) = (G_\mu(z))^{(-1)} - \frac{1}{z}, \qquad \boxed{R_{\mu_c}(z) = R_{\mu_a}(z) + R_{\mu_b}(z)}.$$

Holds for each $\beta > 0$, but **not** for $\beta = 0$.

[Come back to my talk in two weeks for the critical $\beta N \to \gamma$ regime.]

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