# General beta random matrix theory

## (at MATRIX Institute)

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Lecture 1
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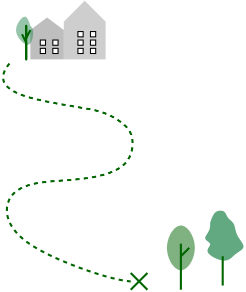
# Roadmap



- What are general $\beta$ random matrices?
- Lecture 1: corners of $\beta$ random matrices.
- Problem set 1.
- Lecture 2: sums of $\beta$ random matrices.
- Problem set 2.
- Lecture 3: questions and discussion of problem sets.

[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

**This is NOT a research talk about brand new results.**
Instead we explore **basic structures and definitions.**

(See “Lattice Paths, Combinatorics and Interactions” in 2 weeks).

# Random matrix theory

The study of **random large** matrices and their eigenvalues.

**Origins:**

- Representation theory of the **classical groups** since 1920s.
  [Groups of matrices come with normalized measures.]
- Multidimensional **statistics** since 1930s.
  [Data is random and is naturally organized in 2-dimensional arrays.]
- Theoretical **physics** since 1950s.
  [Energy levels in heavy nuclei modelled by eigenvalues.]
- **Number theory** since 1970s.
  [Zeros of Riemann zeta-function modelled by eigenvalues.]
- Reemphasized in modern applied and statistical problems.
  ["Big data" revolution.]

The **central** and the **most basic** random matrix object is the **Gaussian Orthogonal/Unitary/Symplectic Ensemble.**

# Gaussian $\beta$ ensembles

$N \times N$ matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

$$M = \frac{X + X^*}{2} = \begin{pmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

The density of **eigenvalues** $x_1 < x_2 < \cdots < x_N$:

$$\sim \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\tfrac{\beta}{4}(x_i)^2\right).$$

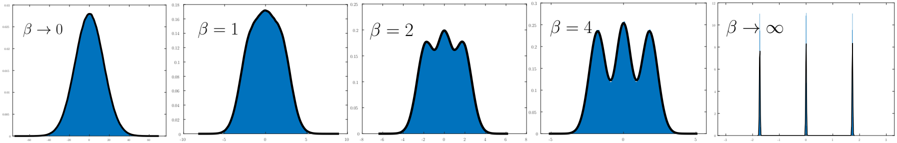
$\beta = 1, 2, 4$ is the **dimension** of the base (skew-) field.

After today's lecture and pset you should be able to **prove it!**

# Gaussian $\beta$ ensembles

$$\prod_{1\leq i<j\leq N}(x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

First correlation function for $N = 3$: $\frac{1}{3}\mathbb{E}\left[\delta_{x_1} + \delta_{x_2} + \delta_{x_3}\right]$




i.i.d. Gaussian | random matrices | Hermite roots

**Five** meaningul values ask for a **unified treatment** of $\beta \in [0, +\infty]$

This is the topic of **general $\beta$ random matrix theory.**

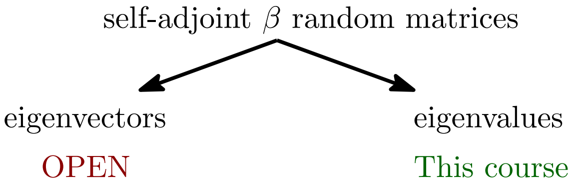
# Tasks of $\beta$ random matrix theory

- **Asymptotic questions:** E.g., $N \to \infty$ behavior of density

$$\prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} V(x_i).$$

with fixed $\beta > 0$, **or** $\beta \to 0$, **or** $\beta \to \infty$.

- **Algebraic questions:**
  How do we add and multiply general $\beta$ random matrices?




*Disclaimer:* There is no field of dimension $\beta$.

**Algebra:** Rank 1 operations as a building block.

# The key computation

$N \times N$ matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables with real parts $\mathcal{N}(0, \frac{2}{\beta})$. $M = \frac{X+X^*}{2}$.

$$\left(\begin{array}{ccc|c} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ \hline M_{41} & M_{42} & M_{43} & M_{44} \end{array}\right)$$

**Eigenvalues:**

- $(\lambda_i)_{i=1}^N$ — $N \times N$
- $(\mu_i)_{i=1}^{N-1}$ — $(N-1) \times (N-1)$

**Theorem.** Conditional distributions are:

1. $(\mu_i)$ given $(\lambda_i)$ solve $$\sum_{i=1}^{N} \frac{\xi_i}{z - \lambda_i} = 0.$$
2. $(\lambda_i)$ given $(\mu_i)$ solve $$\sum_{i=1}^{N-1} \frac{\xi_i'}{z - \mu_i} = z + \mathcal{N}(0, \tfrac{2}{\beta}),$$

$\xi_i$ and $\xi_i'$ are i.i.d. $\frac{1}{\beta}\chi^2_\beta$ random variables, that is $\sum_{j=1}^{\beta} \mathcal{N}_j^2(0, \frac{1}{\beta})$.

**Important:** This is a basis of extension to all $\beta \in [0, +\infty]$.

**Proof that** $(\lambda_i)$ given $(\mu_i)$ solve $\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$ : $I$

## **Proof that** $(\lambda_i)$ given $(\mu_i)$ solve $\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$ : II

## **Proof that** $(\lambda_i)$ given $(\mu_i)$ solve $\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$ : *III*

# Interlacement of eigenvalues

**Corollary 1.** The eigenvalues of a matrix and its corner interlace:

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{N-1} \leq \lambda_N.$$

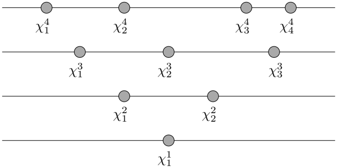
**Proof.**

# **Corollary 2:** The multilevel densities of G$\beta$E

Infinite matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

**All** corners of $M = \frac{X+X^*}{2}$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$




Joint density of **interlacing eigenvalues.**

$$\prod_{k=1}^{N-1} \prod_{1\le i<j\le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1} \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i^N)^2\right)$$

**Gaussian $\beta$ corners process**

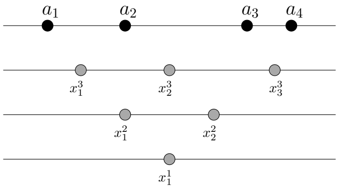
# Corollary 3: $\beta$–corners processes

A self-adjoint matrix $M$ whose law is **invariant** under $M \mapsto UMU^*$

($U$ — orthogonal/unitary/symplectic if $\beta = 1, 2, 4$)

Eigenvalues of corners

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$

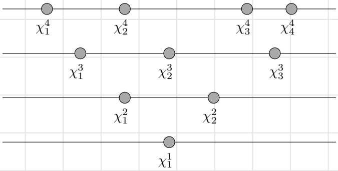



Conditionally on $(x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N)$, the joint law is

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

- A basis of extension from $\beta = 1, 2, 4$ to general $\beta > 0$.
- Consistent with Gaussian $\beta$ Ensembles.

## **Sketch of the proof** for multilevel densities (Corollaries 2 and 3) I

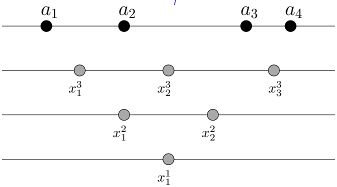


## **Sketch of the proof** for multilevel densities (Corollaries 2 and 3) II

## **Sketch of the proof** for multilevel densities (Corollaries 2 and 3) III

## **Conclusion:** eigenvalues of corners of $\beta$ random matrices

Fix $\beta > 0$
$N = 1, 2, \ldots$
$a_1, \ldots, a_N \in \mathbb{R}$




**Definition.** Eigenvalues of corners of $N \times N$ **random $\beta$-matrix** with uniformly random eigenvectors and fixed eigenvalues $(a_i)_{i=1}^N$ are a triangular array $(x_i^k)_{1 \le i \le N}$ satisfying

$$x_{i+1}^k \le x_i^k \le x_{i+1}^{k+1}, \qquad (x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N),$$

with distribution of density

$$\left[\prod_{k=1}^{N} \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k}\right] \cdot \prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

What about $\beta = 0$ or $\beta = \infty$?

**Theorem.** With $(x_1^N, \dots, x_N^N) = (a_1, \dots, a_N)$, the eigenvalues with law

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

converges as $\beta \to \infty$ to the **roots of derivarives**:

$$\prod_{i=1}^{k} (z - x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N} (z - a_j), \qquad k = 1, 2, \dots, N.$$

**Proof.**

# One asymptotic result

**Theorem.** Suppose that as $N \to \infty$

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i/N} \to \mu, \qquad \text{with} \qquad G_\mu(z) = \int \frac{\mu(dx)}{z-x}$$

$$\prod_{i=1}^{k}(z - x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N}(z - a_j) \qquad \text{and} \qquad k/N \to \alpha.$$

**Then**

$$\frac{1}{k}\sum_{i=1}^{k}\delta_{x_i^k/k} \to \mu_\alpha, \qquad \text{with} \qquad G_{\mu_\alpha}(z) = \int \frac{\mu_\alpha(dx)}{z-x}$$

$$R_\mu(z) = (G_\mu(z))^{(-1)} - \frac{1}{z}, \qquad R_{\mu_\alpha}(z) = (G_{\mu_\alpha}(z))^{(-1)} - \frac{1}{z}$$

$$\boxed{\alpha R_{\mu_\alpha}(z) = R_\mu(z)}.$$

Same result for each $\beta > 0$, but **not** for $\beta = 0$.

End of Lecture 1. Don't forget about Problem set 1.

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