General beta random matrix theory: Problem set 2

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These problems cover and extend the material of Lecture 2. You are very welcome to discuss your solutions, ask questions, or seek for help during the office hours (aka the third lecture). Also feel free to reach out to me at vadicgor@gmail.com for questions or discussions.

Submit your solutions as a single .pdf via e-mail to me (before the end of the summer school) with "Solutions to problem set" in the subject.

Those, who submit solutions to at least one half of the problems of the class, will receive a postcard in the mail. (Please, include your full postal address)

Problem 1. Our aim is to prove the Harish-Chandra/Itzykson–Zuber formula, which evaluates the multivariate Bessel function at $\beta = 2$:

$$B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,1) = \prod_{k=1}^{N-1} k! \cdot \frac{\det\left[e^{a_i z_j}\right]_{i,j=1}^{N}}{\prod_{i<j}(z_i - z_j)(a_i - a_j)}. \tag{1}$$

1. "Branching rule": Using the definition of Lecture 2, prove that for each $\beta > 0$, we have

$$B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,\tfrac{\beta}{2}) = \int_{b_1,\dots,b_{N-1}} e^{z_N\left(\sum_{i=1}^N a_i - \sum_{i=1}^{N-1} b_i\right)}\, B_{b_1,\dots,b_{N-1}}(z_1,\dots,z_{N-1};\,\tfrac{\beta}{2})\, P^{N\to N-1}_{a_1,\dots,a_N;\beta}(db_1,\dots,db_{N-1}),$$

   where the integration goes over all $b_1,\dots,b_{N-1}$ interlacing with $a_1,\dots,a_N$ and $P^{N\to N-1}_{a_1,\dots,a_N;\beta}$ is the conditional probability of Set 1, Problem 4 with $(\lambda_i) = (a_i)$ and $(\mu_i) = (b_i)$.

2. Specializing to $\beta = 2$ and assuming that $a_1,\dots,a_N$ are all distinct, show that

$$B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,1) = (N-1)! \int_{a_1<b_1<a_2<\dots<b_{N-1}<a_N} e^{z_N\left(\sum_{i=1}^N a_i - \sum_{i=1}^{N-1} b_i\right)} \times B_{b_1,\dots,b_{N-1}}(z_1,\dots,z_{N-1};\,1)\, \frac{\prod_{1\le i<j\le N-1}(b_j - b_i)}{\prod_{1\le i<j\le N}(a_j - a_i)}\, db_1\cdots db_{N-1}. \tag{2}$$

3. Show that the right-hand side of (1) satisfies the same recurrence over N as $B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,1)$ in (2).

   Hint: Integrate over $b_1,\dots,b_{N-1}$ under the determinant sign and then manipulate the matrix under the determinant to the desired $B_{a_1,\dots,a_N}(z_1,\dots,z_N;\,1)$.

4. Prove (1) for the case when all a_i are distinct.
5. Show that both sides of (1) are continuous in $a_1, \ldots, a_N$ and deduce that (1) also holds when some of a_i's coincide.

Problem 2. Recall the definition of the Dunkl operator at $\beta = 2$:

$$D_i := \frac{\partial}{\partial z_i} + \sum_{j: j \neq i} \frac{1}{z_i - z_j} \circ (1 - s_{i,j}),$$

where the operator $s_{i,j}$ interchanges the variables z_i and z_j (for instance $s_{1,3}[z_1 + (z_2)^2 + (z_3)^3] = (z_1)^3 + (z_2)^2 + z_3$).

- Using (1) show that

$$(D_1 + \cdots + D_N) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1) = (a_1 + \cdots + a_N) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1).$$

- Show that

$$(D_1^2 + \cdots + D_N^2) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1) = (a_1^2 + \cdots + a_N^2) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1).$$

- Show that[1]

$$(D_1^3 + \cdots + D_N^3) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1) = (a_1^3 + \cdots + a_N^3) B_{a_1,\ldots,a_N}(z_1, \ldots, z_N; 1).$$

[1]This one is harder than the previous two.