General beta random matrix theory: Problem set 1

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These problems cover and extend the material of Lecture 1. You are very welcome to discuss your solutions, ask questions, or seek for help during the office hours (aka the third lecture). Also feel free to reach out to me at vadicgor@gmail.com for questions or discussions.

Submit your solutions as a single .pdf via e-mail to me (before the end of the summer school) with "Solutions to problem set" in the subject.

Those, who submit solutions to at least one half of the problems of the class, will receive a postcard in the mail. (Please, include your full postal address)

Problem 1. Take a self-adjoint matrix $A = [A_{ij}]_{i,j=1}^N$ in N–dimensional space (can be real or complex) with coordinates $(x_1, \dots, x_N)$ and a hyperplane L given by the equation

$$\ell_1 \bar{x}_1 + \ell_2 \bar{x}_2 + \cdots + \ell_N \bar{x}_N = 0.$$

L is a linear subspace and we can restrict A onto it, resulting in A_L.[1] Show that the eigenvalues of A_L are roots of a degree $N-1$ polynomial equation:

$$\det \begin{pmatrix} A_{11} - z & A_{12} & \dots & A_{1N} & \ell_1 \\ A_{21} & A_{22} - z & \dots & A_{2N} & \ell_2 \\ \vdots & & \ddots & & \vdots \\ A_{N1} & & \dots & A_{NN} - z & \ell_N \\ \bar{\ell}_1 & \bar{\ell}_2 & \dots & \bar{\ell}_N & 0 \end{pmatrix} = 0.$$

Problem 2. Let Λ be a *diagonal* $N \times N$ matrix with real eigenvalues $\Lambda_{ii} = \lambda_i$ (might be deterministic or random). Also let A be $N \times N$ random real symmetric (or complex Hermitian) matrix with the same eigenvalues as Λ. Assume that the law of A is invariant under conjugations with orthogonal (or unitary) matrices, i.e. under transformations $A \mapsto UAU^*$ with orthogonal (or unitary) U. Further, let L' be an arbitrary deterministic hyperplane, and let L be a hyperplane orthogonal to a uniformly random unit vector in N–dimensional real (or complex) space. Assume additionally that L and Λ are independent. Show that the eigenvalues of the restrictions Λ_L and $A_{L'}$ have the same probability distributions.

Problem 3. Let A be $N \times N$ random self-adjoint matrix, whose law is invariant under conjugations with orthogonal (or unitary in the complex case) matrices. Let B be the top–left

[1]There are two equivalent ways to define this restriction: you could either identify A with a quadratic form, and then restrict this form onto the subspace L. Or you can think about A as an operator, take an orthogonal projector P onto L and consider PAP as a linear operator operator on L.

$(N-1)\times(N-1)$ corner of A. Show that given the eigenvalues $(\lambda_i)_{i=1}^N$ of A, the eigenvalues $(\mu_i)_{i=1}^{N-1}$ of B can be found as roots of the equation

$$\prod_{i=1}^{N}(z-\lambda_i)\sum_{i=1}^{N}\frac{\xi_i}{z-\lambda_i}=0, \tag{1}$$

where ξ_i are i.i.d. χ^2_β random variables ($\beta=1,2$ depending on whether we deal with real or complex random variables), which are particular cases of the Gamma-distribution with density

$$\frac{1}{2^{\beta/2}\Gamma(\beta/2)}x^{\beta/2-1}e^{-x/2}, \quad x>0.$$

Problem 4. Given $(\lambda_i)_{i=1}^N$ find the probability distribution of the roots of (1).
Hint: Set $w_i=\frac{\xi_i}{\sum \xi_j}$. If μ_j are the roots of (1), then we have

$$\prod_{i=1}^{N}(z-\lambda_i)\sum_{i=1}^{N}\frac{w_i}{z-\lambda_i}=\prod_{j=1}^{N-1}(z-\mu_j), \tag{2}$$

Plugging $z=\lambda_i$, we get an expression for w_i in terms of μ_j. It remains to rewrite the joint density of w_i in terms of μ_j. Don't forget to multiply by the Jacobian of the transformation![2]

Problem 5. Given real numbers $(\mu_i)_{i=1}^{N-1}$ find the probability distribution of the roots of

$$\sum_{i=1}^{N-1}\frac{\xi_i'}{z-\mu_i}=z+\zeta,$$

where ξ_i' are independent $\frac{1}{\beta}\chi^2_\beta$ random variables and ζ is (independent) Gaussian $\mathcal{N}(0,\frac{2}{\beta})$.

Problem 6. Let A be $N\times N$ tridiagonal real symmetric matrix with independent matrix elements (on and above diagonal) of the form (note χ rather than χ^2 random variables!):

$$\begin{pmatrix}
\mathcal{N}(0,\frac{2}{\beta}) & \frac{1}{\sqrt{\beta}}\chi_{(N-1)\beta} & 0 & \dots & 0\\
\frac{1}{\sqrt{\beta}}\chi_{(N-1)\beta} & \mathcal{N}(0,\frac{2}{\beta}) & \frac{1}{\sqrt{\beta}}\chi_{(N-2)\beta} & 0 & \vdots\\
0 & \frac{1}{\sqrt{\beta}}\chi_{(N-2)\beta} & & & \\
\vdots & & \ddots & & \frac{1}{\sqrt{\beta}}\chi_{\beta}\\
0 & \dots & & \frac{1}{\sqrt{\beta}}\chi_{\beta} & \mathcal{N}(0,\frac{2}{\beta})
\end{pmatrix},$$

and let B be *bottom-right* $(N-1)\times(N-1)$ submatrix of A. Prove that the conditional laws of eigenvalues of B given eigenvalues of A and vice versa are given by Problems 4 and 5, respectively.[3]

[2]If you don't make mistakes, then this Jacobian should be given by the Cauchy determinant.

[3]For $\beta=1,2$ this is not hard: all you need to do is to apply some conjugations to GOE/GUE matrices. But for general $\beta>0$ the computation is challenging, and you may find lemmas about tridiagonal matrices from sections 2.2-2.3 of `https://arxiv.org/pdf/math-ph/0206043.pdf` helpful.