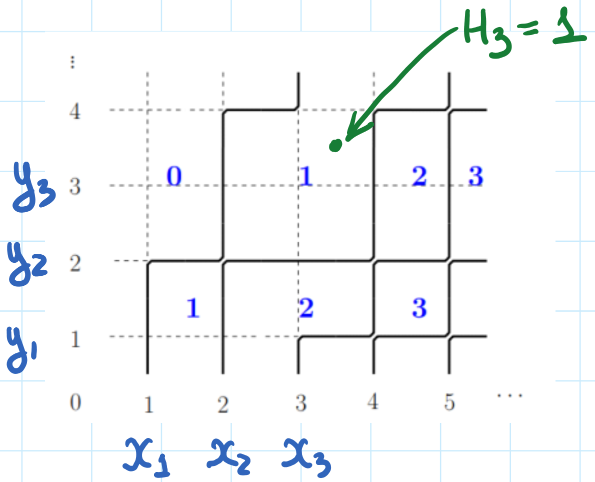
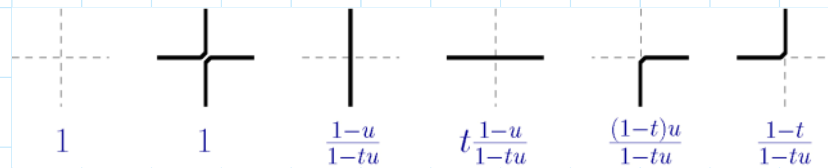
Lecture 5: Schur measures and observables

Last time:

$0<t<1$





$u = x_i y_j$

$$\boxed{\begin{gathered}\mathbb{E}_{6V}\left((1-w)(1-wt)\cdots(1-wt^{H_N-1})\right)\\ \| \\ \mathbb{E}_{Schur}\prod_{i=1}^{N}\left(1-w\,t^{\lambda_i+N-i}\right)\end{gathered}}$$

$$P_{schur}(\lambda) = \prod_{i,j=1}^{N}(1-x_i y_j)\, s_\lambda(x_1,\dots x_N)\, s_\lambda(y_1,\dots y_N)$$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0)$

Corollary (Borodin-18) Let $\ell(\lambda)$ denote # of non-zero coordinates in λ. Assume that for Schur measure

$$\lim_{N\to\infty} \frac{\overset{\#\text{zeros}}{(N-\ell(\lambda))} - d_N}{B_N} \overset{d}{=} \xi$$

← some r.v. with cont. distr.

B_N ← some constants with $B_N \to \infty$ as $N\to\infty$

Then also
$$\lim_{N\to\infty}\frac{H_N - d_N}{B_N} \overset{d}{=} \xi$$

$\boxed{\text{No dependence on } t \text{ here!}}$

Proof sketch: Set $w = -t^{-z_N}$ and divide by $\prod_{i\geq 0}(1+t^{-z_N+i})$

$$\mathbb{E}_{6V}\prod_{j\geq 0}\frac{1}{(1+t^{-z_N+H_N+j})} = \mathbb{E}_{Schur}\prod_{j\in\mathbb{Z}_{\geq 0}\setminus\{\lambda_i+N-i\}}\frac{1}{1+t^{-z_N+j}}$$

some constants

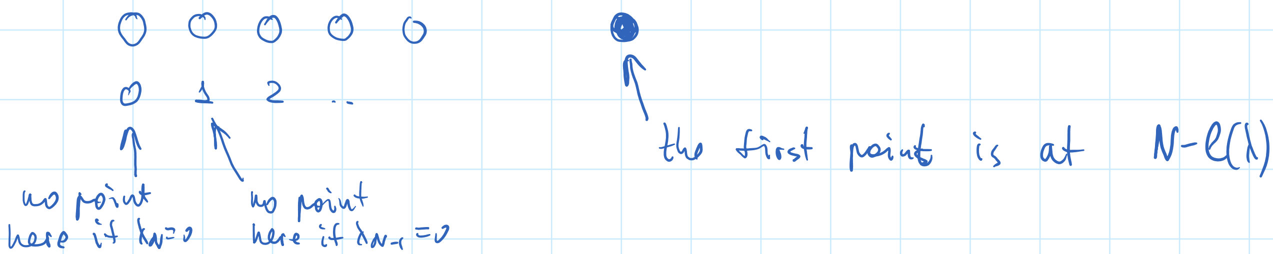
1) LHS $\approx$ Prob$(H_N - z_N \to +\infty)$

If $H_N - z_N \to +\infty$, all factors are close to 1 $\Rightarrow$ observable under $\mathbb{E}_{6v}$ converges to 1

If $H_N - z_N \to -\infty$, then some factors are close to 0, while all of them are between 0 and 1 $\Rightarrow$ observable converges to 0.

2) RHS $\approx \mathrm{Prob}\left(N - \ell(\lambda) - z_N \to +\infty \right)$

Points of $\mathbb{Z}_{\geq 0} \setminus \{\lambda_i + N - i\}$



If $N - \ell(\lambda) \to +\infty$, then all factors under $\mathbb{E}_{\text{Schur}}$ are close to 1, and we approximate indicator

If $N - \ell(\lambda) \to -\infty$, then there is a close to 0 factor $\Rightarrow$ expression under $\mathbb{E}_{\text{Schur}}$ goes to 0

Hence $\mathbb{E}_{6v} = \mathbb{E}_{\text{Schur}}$ identifies asymptotic probabilities for convergence in distribution $\blacksquare$

<u>Conlusion</u>: Need to stydy $\ell(\lambda)$ for

Schur measure - distributed λ.

Classical technology to do this

Determinantal point processes

arXiv > math-ph > arXiv:math-ph/0510038

Mathematical Physics

[Submitted on 10 Oct 2005]

Random matrices and determinantal processes

Kurt Johansson

We survey recent results on determinantal processes, random growth, random tilings and their relation to random matrix theory.

Comments: 40 pages. Lectures given at the summer school on Mathematical statistical mechanics in July 05 at Ecole de Physique, Les Houches

arXiv > math > arXiv:1212.3351

Mathematics > Probability

[Submitted on 13 Dec 2012 (v1), last revised 18 Jan 2015 (this version, v2)]

Lectures on integrable probability

Alexei Borodin, Vadim Gorin

These are lecture notes for a mini-course given at the St. Petersburg School in Probability and Statistical Physics in June 2012. Topics include integrable models of random growth, determinantal point processes, Schur processes and Markov dynamics on them, Macdonald processes and their application to asymptotics of directed polymers in random media.

arXiv > math > arXiv:math/0309074

Mathematics > Combinatorics

[Submitted on 4 Sep 2003]

Symmetric functions and random partitions

Andrei Okounkov

These are (not updated) notes from the lectures I gave at the NATO ASI ``Symmetric Functions 2001'' at the Isaac Newton Institute in Cambridge (June 25 -- July 6, 2001). Their goal is an informal introduction to asymptotic combinatorics related to partitions.

Instead, I give other technology, see [Ahn-20]

Proposition Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$ distributed as Schur meas.

$$P(\lambda) = \prod_{i,j \geq 1}^{N} (1 - x_i y_j) \cdot s_\lambda(x_1, \dots x_N)\, s_\lambda(y_1, \dots y_N)$$

Then for any $0 < q < 1$ we have

$$\mathbb{E} \sum_{j \in \mathbb{Z}_{\geq 0} \setminus \{\lambda_i + N - i\}_{i=1}^{N}} q^j = \frac{1}{2\pi i (1-q)} \oint_{\text{around } \{0, x_1 \dots x_N\}} \prod_{j=1}^{N} \left[\frac{qz - x_j}{z - x_j} \cdot \frac{1 - y_j z}{1 - q y_j z} \right] \frac{dz}{z}$$

Remark There is also k-fold contour integral for $\mathbb{E}$ (product of for $q = q_1, \dots q_k$)

Proof

$$\sum_\lambda \underbrace{s_\lambda(x_1, \dots x_N)}_{\det[x_i^{\lambda_j + N - j}]/\ldots} s_\lambda(y_1, \dots y_N) = \prod_{i,j} (1 - x_i y_j)^{-1}$$

$N(N-1)$ factors $\qquad$ $N-1$ factors

$$\frac{\det\left[x_i^{\lambda_j+N-j}\right]}{\prod_{i<j}(x_i-x_j)}$$

$\frac{N(N-1)}{2}$ factors $N-1$ factors

Apply $$\mathcal{D}_q = \prod_{i<j}(x_i-x_j)^{-1}\left[\sum_{i=1}^{N} T_{q,i}\right]\prod_{i<j}(x_i-x_j) = \sum_{i=1}^{N}\prod_{j\neq i}\frac{qx_i-x_j}{x_i-x_j}\,T_{q,i}$$

$$T_{q,i}\, f(x_1,\dots x_N) = f(x_1,\dots x_{i-1}, q x_i, x_{i+1},\dots x_N)$$

LHS becomes $$\sum_{\lambda}\left(\sum_{i=1}^{N} q^{\lambda_i+N-i}\right) s_\lambda(x_1,\dots x_N)\, s_\lambda(y_1,\dots y_N)$$

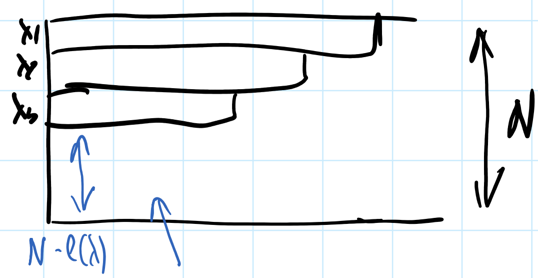
RHS becomes $$\prod_{i,j}(1-x_iy_j)^{-1}\sum_{i=1}^{N}\left[\prod_{j\neq i}\frac{qx_i-x_j}{x_i-x_j}\right]\prod_{j=1}^{N}\frac{1-x_iy_j}{1-qx_iy_j}$$

$$=\prod_{i,j}(1-x_iy_j)^{-1}\frac{1}{2\pi i}\oint_{\{x_i\}}\prod_{j=1}^{N}\left[\frac{qz-x_j}{z-x_j}\,\frac{1-y_jz}{1-qy_jz}\right]\frac{dz}{(q-1)z}$$

Deform contour through 0, collecting residue there and multiply by $\prod_{i,j}(1-x_iy_j)$ ∎

How is this helpful?

$$\sum_{j\in\mathbb{Z}_{\geq 0}\setminus\{\lambda_i+N-i\}} q^j = q^{N-\ell(\lambda)} + \dots$$



Next terms correspond to next columns

For $0<q<1$, the first term is the largest, asymptotic behavior knows about it

Suppose $$N-\ell(\lambda) = N\alpha + N^{1/3}\xi + \bar{o}\left(N^{1/3}\right) \quad \text{as } N\to\infty$$

some random variable

Then for $q = 1 + sN$ ↖ some parameter (real number <0)

$$\sum_{j \in ..} q^j = (1+sN^{-1/3})^{dN} \left(e^{s\xi} + \dots \right)$$

So $\mathbb{E}(\dots)$ is a version of Laplace transform

Thus, probabilistic theorem reduces to studying limit of $\mathbb{E}(..)$

Theorem: Suppose $x_i = 1$, $y_i = u$, $q = 1 + sN^{-1/3}$

Then as $N \to \infty$ for each $s < 0$

$$\mathbb{E} \sum_{j \in \mathbb{Z}_{\geq 0} \setminus \{\lambda_i + N - i\}} q^j \cdot (1+sN^{-1/3})^{-dN} \longrightarrow \frac{1}{2\sqrt{\pi}} (s\sigma)^{-3/2} \exp\left(\frac{(s\sigma)^3}{12} \right)$$

$$d = \frac{1-\sqrt{u}}{1+\sqrt{u}} \qquad \sigma = u^{1/6} \frac{(1-\sqrt{u})^{1/3}}{1+\sqrt{u}}$$

These are scalings of Theorem 2 in Lecture 4

Remark: This only shows that scalings are correct. In order to identify with TW_2, you need to use previous remark about k-fold integrals, pass to the limit there as well, and then repeat on the RM side for the same answer.

Proof of the theorem: Asymptotic analysis of the contour integral in proposition. Formally this is called „steepest descent method", which in this case boils down to deforming the contour to pass through $z = -\frac{1}{\sqrt{u}}$ and then Taylor expanding the integrand near this point. You arrive at a Gaussian integral asymptotically, which gives the answer.

Exercise: Finish this proof!