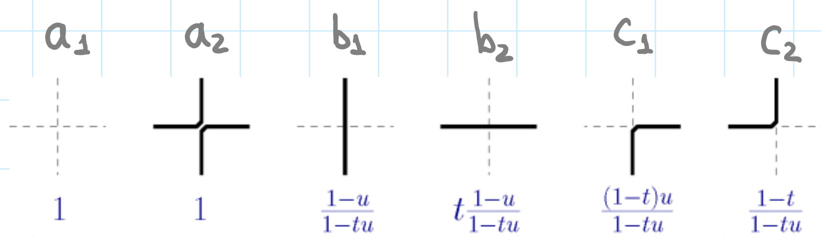
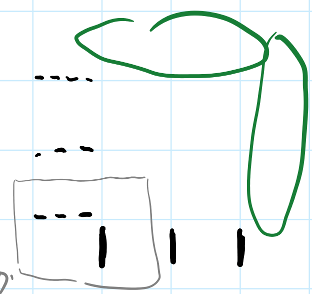
Lecture 4: Stochastic 6v model



$\boxed{\Delta > 1}$

Previously $|t| = |u| = 1$ Today $0<t<1$ $0<u<1$

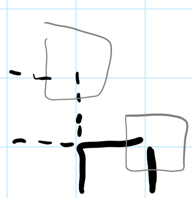
For semi-free boundary conditions we can sample nicely

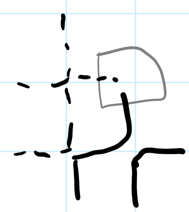


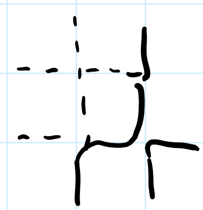
On green boundaries we have no restrictions (in this Lecture)

Sample recurrently: Flip a biased coin and sample

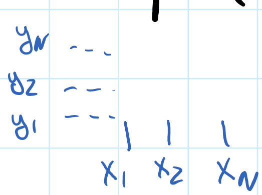
Choose with probabilities b_1/c_1 which configuration to have here







Theorem 1: Let H_N be the value of height function at (N,N) = # paths which exit $N \times N$ square along its top (rather than right) boundary.



$u = x_i y_j$

Then for each $w \in \mathbb{C}$

$$(*)\quad \mathbb{E}\left((1-w)(1-wt)\cdots(1-wt^{H_N-1})\right) = \frac{\prod_{i,j}(1-x_i y_j)}{\prod_{i<j}(x_i-x_j)(y_i-y_j)} \det\left[\frac{1-w-(t-w)x_i y_j}{(1-x_i y_j)(1-t x_i y_j)}\right]$$

$N=1$: LHS = Prob($H_1 = 0$)   RHS = 1×1 determinant

$w=1$: LHS $= \mathrm{Prob}(H_N = 0)$ RHS $=$ JK determinant

Proof from [Aggarwal-Borodin-Wheeler-21], earlier forms [Borodin-18], homogeneous [Borodin-Corwin-G.-16]

Proof follows JK-strategy

Both sides of $(*)$ satisfy:

1) Symmetric in $(x_1, \dots x_N)$ and $(y_1, \dots y_N)$

2) $(*) \cdot \prod_{i,j} (1 - t x_i y_j)$ = Polynomial in x_i, y_j, degree of each variable $\le N$

3) If $x_N = \frac{1}{y_N} \to Z_N(x_1, \dots x_N; y_1 \dots y_N; t, w) = Z_{N-1}(x_1, \dots x_{N-1}; y_1, \dots y_{N-1}; t, w)$

4) $Z_N(0, \dots 0; y_1, \dots y_N; t, w) = (1-w)(1-wt) \cdot \dots \cdot (1 - wt^{N-1})$.

Checking 1,2,3 for LHS and RHS is as before in Lecture 2

Property 4 for LHS: all paths are vertical $|||$ $\Rightarrow H_N = N$

Property 4 for RHS: later

Why do 1-4 uniquely determine LHS, RHS?

Induction in N. $N=1 \to$ exercise

Step As a polynomial in x_N

$$\prod (1 - t x_i y_j) Z_N^{RHS}(x_1 \dots x_N; y_1, \dots y_N; t, w) =$$

$\parallel$ in N points $\frac{1}{y_i}$

$$\prod (1 - t x_i y_j) Z_N^{LHS}(x_1 \dots x_N; y_1 \dots y_N; t, w)$$

Thus,
$$\prod (1 - t x_i y_j) Z_N^{RHS} = \prod (1 - t x_i y_j) Z_N^{LHS} + \underbrace{(\quad)}_{\substack{\text{should} \\ \text{not depend on } x_N}} \prod_{j=1}^{N} \left(x_N - \frac{1}{y_j}\right)$$

By symmetry, same is true for x_N replaced by x_i

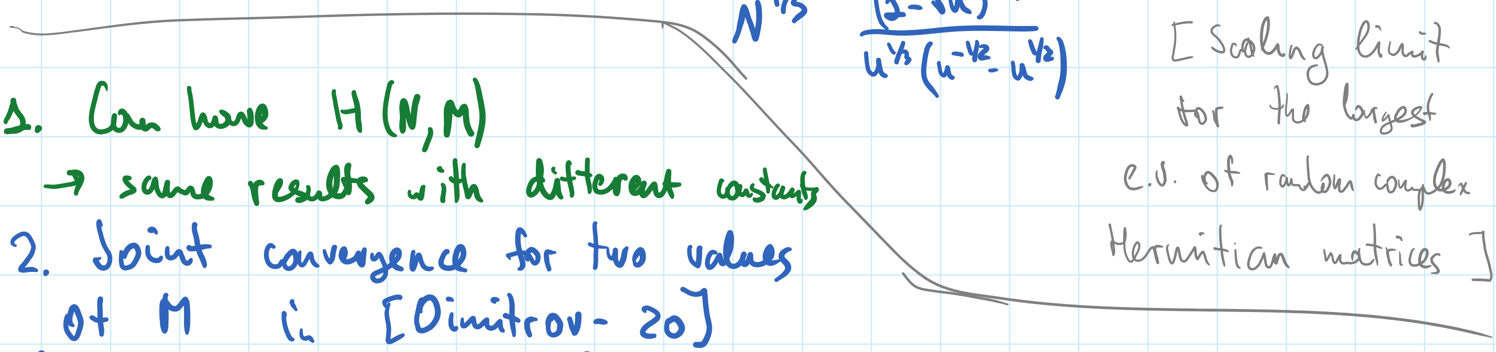
Therefore, $\prod(1-tx_i y_j)\, Z_N^{RHS} = \prod(1-tx_i y_j)\, Z_N^{LHS} + (\ \ )\prod_{i,j}\left(x_i - \frac{1}{y_j}\right)$

Plug $x_1 = x_2 = \ldots = 0 \Rightarrow$ this constant by comparing the degrees is 0. ∎

Theorem 2 (Borodin-Corwin-G.-16) Assume that all x_i are equal and all y_j are equal, so that $0<t<1$, $0<u=x_i y_j<1$.

Then, in distribution
$$\frac{H_N - N\left(\frac{1-\sqrt{u}}{1+\sqrt{u}}\right)}{N^{1/3}\ \frac{(1-\sqrt{u})^{4/3}}{u^{1/3}(u^{-1/2}-u^{1/2})}} \xrightarrow{N\to\infty} TW_2$$

[Scaling limit for the largest e.v. of random complex Hermitian matrices]



1. Can have $H(N,M)$
   → same results with different constants
2. Joint convergence for two values of M in [Dimitrov-20]
   (General multipoint is open)
3. This is for $\Delta>1$. If $-1<\Delta<1$, then height function is expected to converge to Gaussian Free Field
   For $\Delta<-1$: either GFF or special „gas" / „smooth" region, where fluctuations are very small

Our proof is based on:

Proposition: RHS of (*) is also
$$\sum_{\lambda_1\geq\ldots\geq\lambda_N\geq 0}\ \prod_{i=1}^{N}\left(1-wt^{\lambda_i+N-i}\right)\cdot \underbrace{\frac{S_\lambda(x_1,\ldots x_N)\, S_\lambda(y_1,\ldots y_N)}{\prod_{i,j}(1-x_i y_j)^{-1}}}_{\text{„Schur measure"}}$$
$$\| $$
$$\mathbb{E}_{\text{random }\lambda}\prod_{i=1}^{N}\left(1-wt^{\lambda_i+N-i}\right)$$

$S_\lambda(x_1,\ldots x_N) = \dfrac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^{N}}{\ldots}$ Schur symmetric

$$s_\lambda(x_1, \dots, x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^N}{\prod_{i<j} (x_i - x_j)}$$ Schur symmetric polynomials

[homogeneous of degree $\lambda_1 + \dots + \lambda_N$]

Proof

$$\det\left[\frac{1-w-(t-w)x_iy_j}{(1-x_iy_j)(1-tx_iy_j)}\right] = \det\left[\frac{1}{1-x_iy_j} - \frac{w}{1-tx_iy_j}\right]$$ geometric series

$$= \det\left[\sum_{a=0}^{\infty} (1-wt^a)(x_iy_j)^a\right] = \sum_{0 \le a_1 < \dots < a_N} \prod_{i=1}^{N} (1-wt^{a_i}) \det\left[x_i^{a_j}\right] \det\left[y_i^{a_j}\right]$$

By Cauchy-Binet — computation of $\det[P\,Q]$ (rectangular matrices) ∎

Corollary If we plug $x_1 = x_2 = \dots = x_N = 0$, only one term survives: $\lambda_1 = \lambda_2 = \dots = \lambda_N = 0$ and we get $\prod_{i=1}^{N}(1-wt^{N-i})$, as promised in Property 4 inside the proof of Theorem 1.