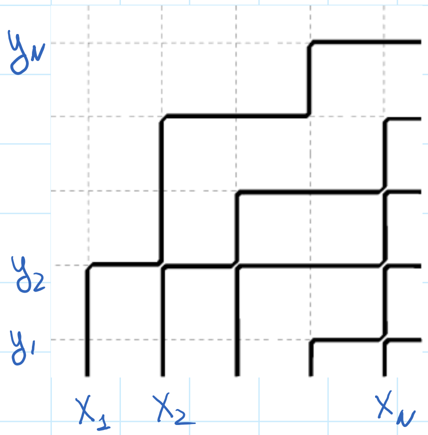
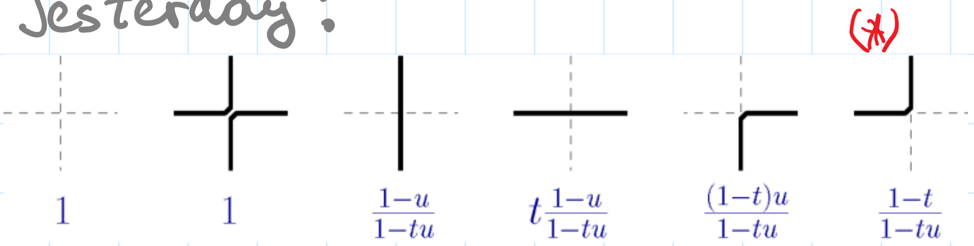
Lecture 3: GUE-Corners process in 6v model with DWBC at Δ=0



Yesterday:



$u = x_i y_j$

Theorem: $Z(x_1,\dots x_N; y_1,\dots y_N; t) = \frac{\prod_{i,j=1}^{N}(1-x_iy_j)}{\prod_{i<j}(x_i-x_j)(y_i-y_j)} \det\left[\frac{(1-t)\,x_iy_j}{(1-x_iy_j)(1-t\,x_iy_j)}\right]$ (**)

Today: Probabilistic consequence $t=-1$

Theorem 2: $t=-1$, $x_i = u$, $y_i = 1$ $|u|=1$, $u\neq -1$

Consider random configuration with DWBC in $N\times N$ square. Let ξ_i^j be the position of the i-th  in j-th row

Then as $N\to\infty$:

1) Prob(there are j  in j-th row) $\to 1$

2) $$\lim_{N\to\infty}\left\{\frac{\xi_i^j-\gamma N}{\sqrt{\gamma(1-\gamma)}\,\sqrt{N}}\right\}_{1\le i\le j} \xrightarrow{\text{in finite-dim distribution}} \left\{\lambda_i^j\right\}$$ GUE-corners process of Lecture 1

$\gamma = \frac{1-u}{2}\cdot\frac{1-u^{-1}}{2}$ $\quad 0<\gamma<1$ $\quad u=\sqrt{-1} \Rightarrow$ J-N theorem from Lecture 1

Proof step 1: What special at $t=-1$?

Lemma (Cauchy determinant)

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$$\det\left[\frac{1}{a_i-b_j}\right]_{i,j=1}^{N} = \frac{\prod_{i<j}(a_i-a_j)(b_j-b_i)}{\prod_{i,j}(a_i-b_j)}$$

Corollary 1: $Z(x_1,\dots x_N; y_1,\dots y_N; -1) = \prod_{i=1}^{N}(2x_iy_i)\cdot\frac{\prod_{i<j}(x_i+x_j)(y_i+y_j)}{\prod_{i,j=1}^{N}(1+x_iy_j)}$

Proof) $(1-x_iy_j)(1-t\,x_iy_j)\Big|^{t=-1} = 1-x_i^2y_j^2 = y_j^2(y_j^{-2}-x_i^2)$

apply Lemma. ∎

Proof step 2 From now on, we deal with $1\le i\le j\le \underline{\underline{2}}$

general case is the same

Proposition: Take $z_1, z_2\in\mathbb{C}$

$$(*)\quad \frac{Z(u^N; e^{iz_1/\sqrt{N}}, e^{iz_2/\sqrt{N}}, 1^{N-2}; -1)}{Z(u^N; 1^N; -1)} \overset{N\to\infty}{=}$$

$$\exp\left(\sqrt{N}\; i(z_1+z_2)\left(\frac{1-u}{2(1+u)}\right) - \frac{z_1^2+z_2^2}{2}\left(\frac{(1-u)^2}{4(1+u)^2}\right) + \bar{o}(1)\right)$$

Proof) By Corollary 1 and a computation ∎

Proof step 3) Interpret probabilistically

Definition: $(\# b \text{ vertices in row } j) = \gamma N + \sqrt{N}\,\eta_j$

Corollary 2: $(\eta_1, \eta_2) \xrightarrow{N\to\infty} \text{i.i.d. } N(0, \gamma(1-\gamma))$

$[\; u = \sqrt{-1} \rightarrow \gamma = \frac{1}{2},\; \sigma^2 = \frac{1}{4} \;]$

Proof:

$$(*) = \frac{\sum_{\text{configurations}} \text{weight}}{\sum_{\text{configurations}} \text{weight}'} = \underbrace{\mathbb{E}_{\text{weight}'}}_{\text{prob. measure}} \underbrace{\frac{\text{weight}}{\text{weight}'}}_{\substack{\text{positive} \\ \text{random variable}}}$$

from theorem

$$= \mathbb{E}\left[\prod_{j=1}^{2} \left(\frac{1 - e^{i z_j/\sqrt{N}} u}{1 - u} \cdot \frac{1+u}{1+e^{i z_j/\sqrt{N}} u} \right)^{\# b\text{-vertices in line } j} \times \underbrace{\left(1 + O(N^{-1/2})\right)^{\# c\text{-vertices in line } j}}_{\text{negligible asymptotically}} \right]$$

$(**)$

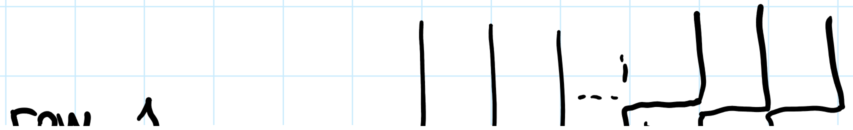
Choose γ, so that $\exp((..)\sqrt{N})$ factor cancels between $(**)$ and $(*)$. In subleading term, you get

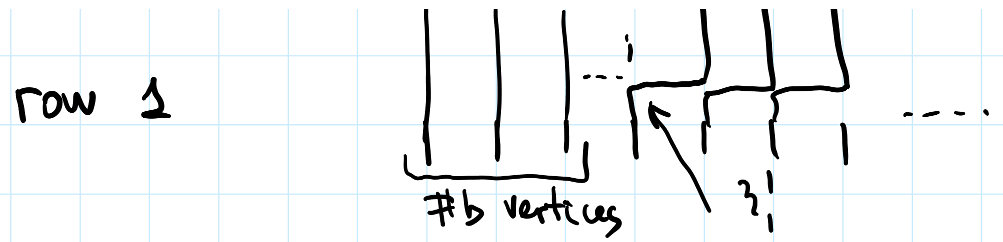
$$\mathbb{E} \exp\left((\eta_1 z_1 + \eta_2 z_2) \cdot \frac{(-2iu)}{1-u^2} \right) \xrightarrow{N\to\infty} \exp\left(\left(\frac{z_1^2}{2} + \frac{z_2^2}{2} \right) \cdot \frac{1}{4} \right)$$

Laplace transform of (u_1, η_2) → Laplace transform of i.i.d. Gaussian random variables ∎

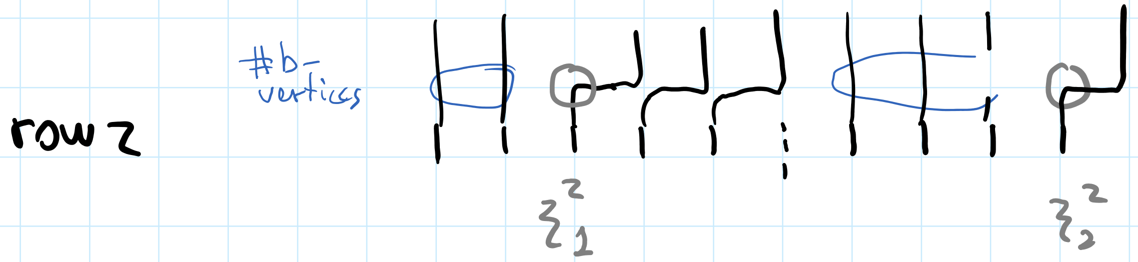
Proof Step 4: Identify with GUE

How is # b- vertices related to ξ_i^j (6v side)





$\xi_1^1 \approx$ #b vertices in row 1



$(\xi_1^2+\xi_2^2) - \xi_1^1 \approx$ # b vertices in row 2

GUE-side

$$\begin{pmatrix}\boxed{m_{11}} & m_{12}\\ m_{21} & m_{22}\end{pmatrix}$$

λ^1 - 1x1 corner

λ_1^2 λ_2^2 - 2 eigenvalues of 2x2 corner

$(\lambda_1^2+\lambda_2^2) - \lambda_1^1 = m_{22}$ - independent and same varianc as $m_{11} = \lambda_1^1$

Proof Step 5: This 2d marginal uniquely determines the full law of 3d random vector

1) On 6v side - Gibbs property

Given ξ_1^2 and ξ_2^2, we know the law of ξ_1^1

As $N\to\infty$ ξ_1^1 is uniformly distributed between ξ_1^2 and ξ_2^2

2) On GUE - side there is the same Gibbs property (proof omitted) coming from the invariance under unitary conjugations: Conditional on λ_2^2, λ_1^2, λ_1^1 is a uniform point between them

$$\mathbb{E}\exp\Big(\mathrm{Trace}\big(GUE\cdot \underset{\substack{\text{matrix}\\ \text{of parameters}}}{\underset{\uparrow}{Z}}\big)\Big) \overset{\substack{\text{unit.}\\ \text{inv.}}}{=} \mathbb{E}\exp\Big(\mathrm{Trace}\big(GUE\cdot\mathrm{diag}(\underbrace{z_1,z_2}_{\text{eigenvalues of } Z})\big)\Big)$$

$$\|$$

Laplace transform of m_{11}, m_{22}

More details in [G - CMP - 2014] $\blacksquare$