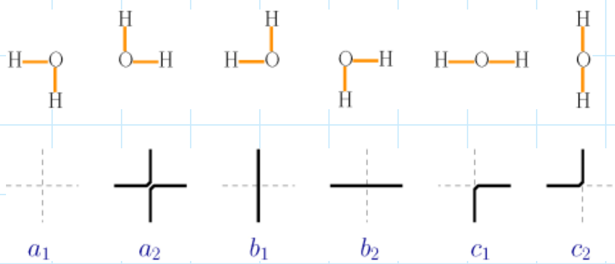
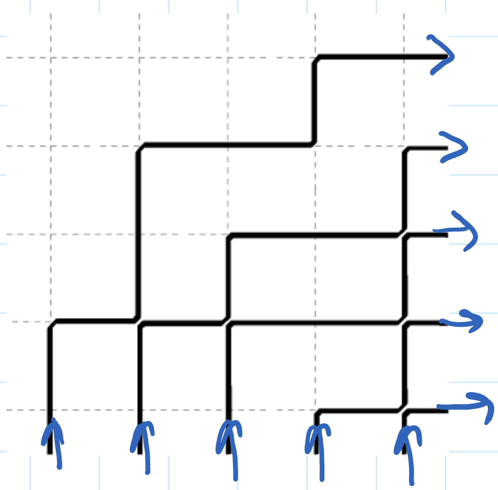
Lecture 2: Izergin-Korepin determinant

Lecture notes on my webpage



bijection

Domain wall boundary conditions



Proposition: If we fix a domain and entrance/exit points for the paths then probability measure
$P(\text{config}) \sim \prod_{\text{vertices}} \text{weight}(\text{vertex})$ depends only on two ratios
$\frac{a_1 a_2}{b_1 b_2}$ and $\frac{a_1 a_2}{c_1 c_2}$ [2 parameters!].

Proof! We find four relations

1) Total # of vertices is fixed

$\# a_1 + \# a_2 + \# b_1 + \# b_2 + \# c_1 + \# c_2 = \text{constant}$

$\Rightarrow$ $P(\text{config}) = \frac{\prod(\text{weights})}{\ldots}$ is unchanged when

$\#a_1 + \#a_2 + \#b_1 + \#b_2 + \#c_1 + \#c_2 = \text{constant}$

$\Rightarrow \quad P(\text{config}) = \dfrac{\Pi(\text{weights})}{\Sigma\,\Pi(\text{weights})}$ is unchanged when we multiply all params by the same number

2) c_1/c_2 vertices change direction of path, but entry/exit directions are fixed.

Hence $\#c_1 - \#c_2 = \text{constant}$

$(c_1, c_2) \to (c_1 \cdot \alpha, c_2 \cdot \frac{1}{\alpha})$ does not change the measure

3) Total # edges occupied by paths is fixed

$2\#a_2 + \#b_1 + \#b_2 + \#c_1 + \#c_2 = \text{constant}$

Combine with 1) to get $\#a_1 - \#a_2 = \text{constant}$ (3')

Hence, $(a_1, a_2) \to (a_1 \cdot \alpha, a_2 \cdot \frac{1}{\alpha})$ does not change the measure

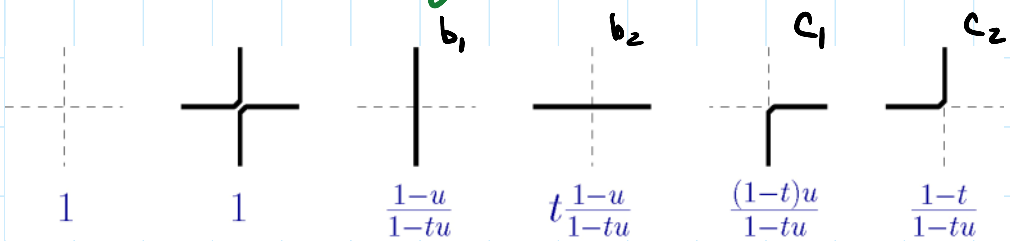
4) Total # vertical edges is fixed

$\#a_2 + \#b_1 + \frac{1}{2}\#c_1 + \frac{1}{2}\#c_2 = \text{constant}$

combine with 3') and then with 1) to get $\#b_1 - \#b_2 = \text{const}$

Hence, $(b_1, b_2) \to (\alpha b_1, \frac{1}{\alpha} b_2)$ does not change the measure.

Our choice of weights: depend on t and u



Features: 1) $b_1 + c_1 = 1$ $\quad b_2 + c_2 = 1$

2) Get positive weights of configurations if either $t > 0,\ u > 0$ or $t \in \mathbb{C}\ |t| = 1,\ u \in \mathbb{C},\ |u| = 1$

$|\Delta| > 1$ $\quad |\Delta| < 1$

For instance, $t = -1$ $\quad \Delta = \dfrac{a_1a_2 + b_1b_2 - c_1c_2}{2\sqrt{a_1a_2b_1b_2}} = 0$

For instance, $t=-1$, $\Delta = \frac{a_1a_2+b_1b_2-c_1c_2}{2\sqrt{a_1a_2b_1b_2}} = 0$,

3) YB relation holds.

Theorem (Izergin-Korepin 82-87)

$u = x_i y_j$

$$Z(x_1,\ldots x_N; y_1,\ldots y_N; t) = \frac{\prod_{i,j=1}^{N}(1-x_iy_j)}{\prod_{i<j}(x_i-x_j)(y_i-y_j)} \det\left[\frac{(1-t)x_iy_j}{(1-x_iy_j)(1-tx_iy_j)}\right]_{i,j=1}^{N}$$

$$\| $$

$$\sum_{\text{config}} \prod(\text{weights})$$

Proof Induction in N

$\underline{N=1}$ LHS =  $= \frac{(1-t)x_1y_1}{1-tx_1y_1}$

RHS $= (1-x_1y_1)\cdot\frac{(1-t)x_1y_1}{(1-x_1y_1)(1-tx_1y_1)}$

Induction step is based on noticing that $\underline{\text{both}}$ LHS and RHS satisfy:

1) Z is symmetric in $(x_1,\ldots x_N)$ and $(y_1, y_2,\ldots y_N)$

2) $Z\cdot\prod_{i,j}(1-tx_iy_j)$ is a polynomial in x_i, y_j with degree of each variable $\leq N$

3) If $x = \frac{1}{\ldots}$ then $Z(x_1, \ldots x_N; y_1, \ldots y_N; t)$

3) If $x_N = \frac{1}{y_N}$, then $Z(x_1, \dots x_N; y_1, \dots y_N; t)$ $=$ $Z(x_1, \dots x_{N-1}; y_1, \dots, y_{N-1}; t)$

4) If $x_i = 0$, then $Z = 0$.

Why do 1–4 imply induction step?

$\prod_{i,j}(1 - x_i y_j) Z(x_1, \dots x_N; y_1, \dots y_N; t)$ is a polynomial in x_N of degree $\le N$ with known values at $(N+1)$ points

$0, \frac{1}{y_1}, \frac{1}{y_2}, \dots, \frac{1}{y_N}$ $\Rightarrow$ it is uniquely determined.

Why does RHS satisfy 1–4?

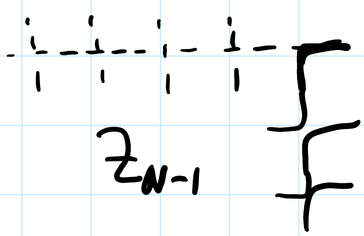
Why does LHS satisfy 1–4?

2) - all the weights become polynomials

3) $u = 1$ $\Rightarrow$ b_1 and b_2 vertices are prohibited

$\Rightarrow$ in top-right corner we have c_1-vertex

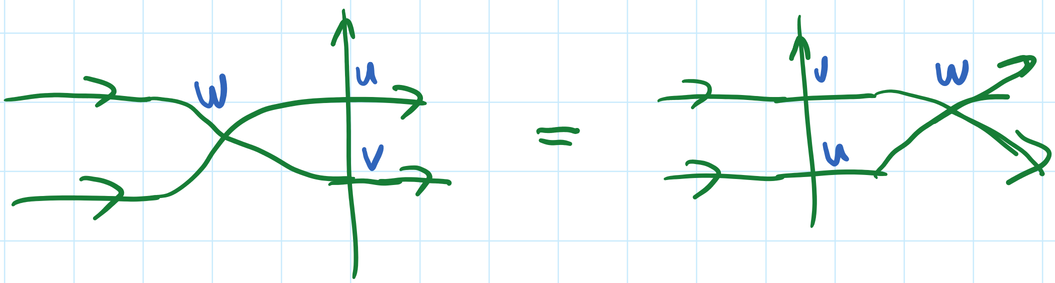
the rest becomes



4) $x_1 = 0$ $\Rightarrow$ $c_1 = 0$ in first column, but there must be a c_1-vertex there $\Rightarrow$ $Z_N = 0$

Theorem 2: (Yang-Baxter relation)

$\frac{v}{u} = w$



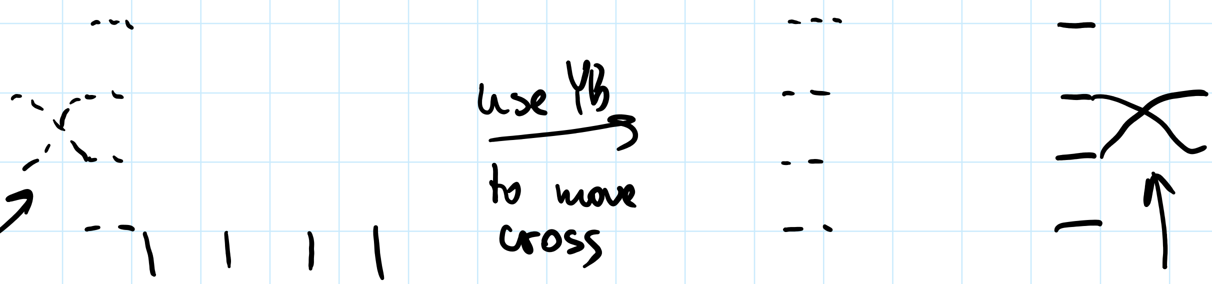
$\frac{v}{u} = w$

[This is u-parameter in my weights]

These are $2^6 = 64$ identities in one picture: specity which outer edges have or do not have paths

<u>Proof</u>: Direch check of 64 cases. ∎

Symmetry of Z in two variables



add one more vertex. No paths $\Rightarrow Z$ is not changing

remove this vertex. Z is not changing ∎.