Lecture 2: Izergin-Korepin determinant

Proposition: If we fix a domain and entrance/exit points for the paths then probability measure \( P(\text{config}) \) depends only on two ratios \( \frac{a_1 a_2}{b_1 b_2} \) and \( \frac{a_1 a_2}{c_1 c_2} \) [2 parameters!].

Proof: We find four relations
1) Total \# of vertices is fixed
\[ a_1 + a_2 + b_1 + b_2 + c_1 + c_2 + b_2 = \text{constant} \]
2) \( \text{O(vert)} \) \( \neq \) \( \text{O(weights)} \) is unchanged when
\[ a_1 + a_2 + b_1 + b_2 + c_1 + b_2 = \text{constant} \]

\[ P(\text{config}) = \frac{\Pi(\text{weights})}{\Sigma \Pi(\text{weights})} \]

is unchanged when we multiply all pairs by the same number.

2) \( c_1/c_2 \) vertices change direction of path, but entry/exit directions stay fixed.

Hence \( c_1 - c_2 = \text{constant} \)

\[ (c_1, c_2) \rightarrow (c_1 \cdot t, c_2 \cdot \frac{1}{t}) \] does not change the measure.

3) Total \# edges occupied by path is fixed

\[ 2\#a_2 + \#b_1 + \#b_2 + \#c_1 + \#c_2 = \text{constant} \]

Combine with 1) to get \( \#a_1 - \#a_2 = \text{constant} \) (3')

Hence, \( (a_1, a_2) \rightarrow (a_1 \cdot t, a_2 \cdot \frac{1}{t}) \) does not change the measure.

4) Total \# vertical edges is fixed

\[ \#a_2 + \#b_1 + \#b_2 + \#c_1 + \frac{1}{t} \#c_2 = \text{constant} \]

Combine with 3') and then with 1) to get \( \#b_1 - \#b_2 \text{ is constant} \)

Hence, \( (b_1, b_2) \rightarrow (tb_1, \frac{1}{t}b_2) \) does not change the measure.

Our choice of weights: depend on \( t \) and \( u \)

\[
\begin{array}{ccc}
\text{Features:} & b_1 + c_1 = 1 & b_2 + c_2 = 1 \\
1 & 1 & 1-u \\
1-t & 1-tu & (1-t)u \\
1-tu & 1-t & 1-tu \\
\end{array}
\]

For instance, \( t = -1 \)

\[
D = \frac{a_1a_2 + b_1b_2 - c_1c_2}{2(a_1b_1, b_1b_2)} = 0
\]
For instance, \( t = -1 \), \( \Delta = \frac{a_1 a_2 + b_1 b_2 - c_1 c_2}{2a_1 a_2 b_1 b_2} = 0 \)

3) YB relation holds.

**Theorem** (Zarembo-Korepin 82-87)

\[
\begin{align*}
\text{...} & \quad \text{...} \\
\beta & = x_1 y_1 \\
\gamma & = x_2 y_2 \\
\eta & = x_1 x_2 \\
\nu & \quad \text{...} \\
\end{align*}
\]

\[
\mathbb{Z}(x_1, x_2, y_1, y_2, \ldots, y_n, t) = \frac{\prod (1-x_i y_j)}{\prod (1-x_i y_j)(y_i y_j)} \det \left[ \frac{(1-t)x_i y_j}{(1-x_i y_j)(1-t x_j y_i)} \right]^{\nu}_{i,j}
\]

\( \sum \) contig \( \prod \) (weights)

**Proof** Induction in \( N \)

\( N=1 \)

\[
\begin{align*}
\text{LHS} & = \frac{1-t}{1-t x_1 y_1} \\
\text{RHS} & = (1-x_1 y_1) \cdot \frac{(1-t)x_1 y_1}{(1-x_1 y_1)(1-t x_1 y_1)}
\end{align*}
\]

Induction step is based on noticing that both LHS and RHS satisfy:

1) \( \mathbb{Z} \) is symmetric in \( (x_1, \ldots, x_n) \) and \( (y_1, y_2, \ldots, y_n) \)

2) \( \mathbb{Z} \cdot \prod (1-t x_i y_j) \) is a polynomial in \( x_i, y_j \) with degree of each variable \( \leq N \)

\[
\mathbb{Z} = 1 - t \ldots \text{...} \ldots \text{...} + \text{...}
\]
3) If \( x_n = \frac{1}{y_n} \), then \( \mathcal{E}(x_1, x_n; y_1, \ldots, y_n; t) \)
\( \mathcal{E}(x_2, \ldots, x_{n-1}; y_2, \ldots, y_{n-1}; t) \)

4) If \( x_i = 0 \), then \( \mathcal{E} = 0 \).

Why do 1-4 imply induction step?
\( \mathcal{E}(x_1, x_n; y_1, \ldots, y_n; t) \) is a polynomial in \( x_n \)
of degree \( \leq N \) with known values at \( (N+1) \) points
\( 0, \frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_N} \) \( \Rightarrow \) it is uniquely determined.

Why does RHS satisfy 1-4?

Why does LHS satisfy 1-4?

2) All the weights become polynomial
3) \( u = \delta \Rightarrow b_1 \) and \( b_2 \) vertices are prohibited
   \( \Rightarrow \) in top-right corner we have \( c_1 \text{-vertex} \)
   the rest becomes
   \[
   Z_{n-1} = \int \int \ldots \int
   \]

4) \( x_i = 0 \Rightarrow c_1 = 0 \) in first column, but there
   must be a \( c_1 \text{-vertex} \) there \( \Rightarrow Z_n = 0 \)

**Theorem 2:** (Young - Pianer relation)
\[
\frac{u}{w} = w
\]
These are \(2^6 = 64\) identities in one picture; specify which outer edges have or do not have paths.

**Proof:** Direct check of 64 cases. \(\Box\)

Symmetry of \(z\) in two variables.

- Use \(y_b\) to move across.
- One more vertex. No paths \(\Rightarrow z\) is not changing.
- Remove this vertex. \(z\) is not changing. \(\Box\).