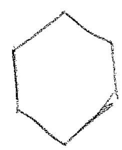
Today we discuss an application of CLT's developped in previous 4 lectures.

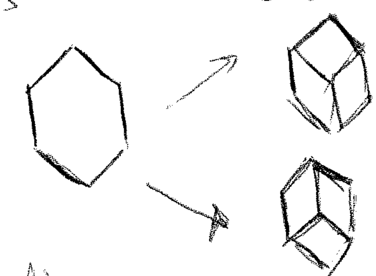
We study uniformly random lozenge/domino tilings of polygonal domains. Handout shows examples.

Phenomenology is similar and we stick to lozenge case.

Consider a polygonal domain on triangular grid.

(E.g. hexagon ), fix mesh size $\frac{1}{L}$,

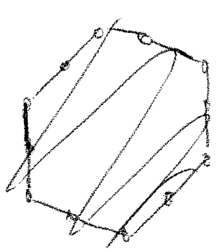
There are finitely many tilings with lozenges

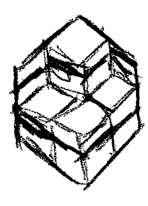
(e.g. )

Consider uniformly random tiling.

Question: How does it look like as $L \to \infty$?

Answer is given in terms of height function





Draw paths 

$H(x,y)$ = number of paths below (x,y).

(up to affine transforms) "=" number of ◇ above (x,y) (or below)

Theorem (Cohn–Kenyon–Propp – 2000)

For any polygon $\lim_{L\to\infty} \frac{H_L(x,y)}{L} = h(x,y)$

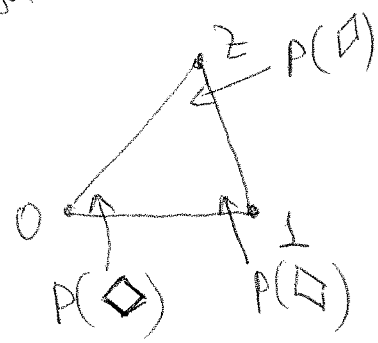
deterministic

$h(x,y)$ is found as a unique maximizer of

$$\iint_{\text{polygon}} \sigma(\nabla h)\, dx\, dy$$ with appropriate boundary conditions.

explicit „surface tension"

Theorem: (Kenyon–Okounkov – 2006). Introduce complex slope $z(x,y)$



$p(◇)$, $p(▱)$, $p(▱)$ — local proportions of 3 types of lozenges = partial derivatives of h in appropriate directions

In „liquid region" all types of lozenges are present and $z \in \mathbb{C}$, i.e. it is non-real.

„Frozen" regions $\to$ real z. „Arctic curve" $\to$ z becomes real

1 type of lozenges.

frozen/liquid border

Then $z(x,y)$ is an algebraic function of (x,y). It is found algorithmically.

(Proven for simply-connected polygons. Believed to be true in general.)

<u>Conjecture</u> (Kenyon-Okounkov)

In the liquid region $\mathcal{L}$ $\lim_{L\to\infty} H_L(x,y) - \mathbb{E} H_L(x,y) = G_z(x,y)$

no scaling!

$=$ The Gaussian Free Field in $\mathcal{L}$ with Dirichlet boundary conditions and in <u>complex structure</u> given by z.

[No fluctuations in frozen regions]

By now some specific cases are settled (Kenyon, Petrov, [Berestycki, Laslier, Ray] Duits, Bufetov-Gorin, Borodin-Ferrari), but in the most general form it is still open

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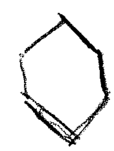
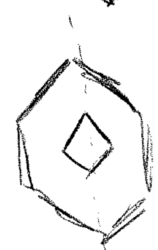
<u>Theorem</u> The conjecture is true for trapezoids (Bufetov-Gorin) and gluings of trapezoids along a single axis (combination of Bufetov-Gorin, Borodin-Gorin-Guionnet, Borot-Gorin-Guionnet).

<u>Examples:</u> Trapezoid —



Hexagon (simply connected)

gluing of 2 trapezoids



Holey Hexagon (topology of annulus)

[Remark: height is <u>fixed</u> along the boundary of the hole]

What is GFF?

A) $\mathbb{H} = \{\operatorname{Im} z > 0\}$ with <u>standard</u> complex structure

$G(\Omega)$ - generalized centered gaussian function

$$\mathbb{E}\, G(\Omega_1)\, G(\Omega_2) = -\frac{1}{2\pi} \ln \left| \frac{\Omega_1 - \Omega_2}{\Omega_1 - \bar{\Omega}_2} \right|$$

Green function of Laplace in $\mathbb{U}$.

The values of G are not defined, however integrals

$$\int_\gamma f(\Omega)\, G(\Omega)\, dl \qquad \int_{\mathcal{D}} f(\Omega)\, G(\Omega)\, dx\, dy$$

for smooth f are bona fide centered gaussians.

B) Simply-connected domain $\mathcal{D}$ with a complex structure $z(x,y)$ (i.e. continuous complex function).

Consider $\Omega : \mathcal{D} \to \mathbb{U}$ – <u>bijection</u>

$(x,y) \to \Omega(x,y)$.

which is <u>conformal</u> w.r.t. $z(x,y)$ [a bit more delicate when z is locally constant]

Riemann uniformization theorem: Ω exists and is unique up to Moebius transformations

Set $G_z(x,y) = G(\Omega(x,y))$

„pullback of GFF in $\mathbb{U}$ by map Ω"

C) A Domain of more complicated topology: either find a green function of Laplace operator defined in a local coordinate system of complex slope

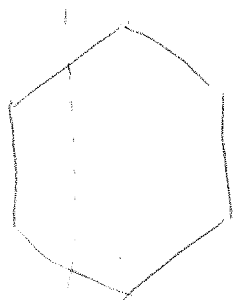
<u>OR</u> map to a subset of plane (conformally) and take a pullback of GFF (= Gaussian with Green function covariance) in this domain.

2) In what topology do we prove convergence?

Values at points make no sense (need different rescaling)

We consider pairings with test functions

$\delta_{x=x_0} \cdot f(y)$ ← polynomial



$$\int \Big(H_L(x_0, y) - \mathbb{E} H_L(x_0, y) \Big) f(y)\, dy \to$$

$$\to \int G\big(\Omega(x_0, y)\big) f(y)\, dy$$

Jointly in finitely many x_0, f.

Such test functions are dense $\Rightarrow$ uniquely define limit.

We will do two things

1) Sketch <u>Heuristic</u> argument for general domains
2) Sketch our rigorous argument.

<u>Heuristics</u>. How is LLN (variational principle) developped?

Take a height function $\tilde{h}(x, y)$

$$P\left(\frac{H_L(x,y)}{L} \in \varepsilon\text{-neighborhood of } \tilde{h}(x,y) \right) \approx \exp\left(L^2 \iint \sigma(\nabla \tilde{h})\, dx\, dy + \bar{\bar{o}}(L^2) \right)$$

↑ decays as $\varepsilon \to 0$

where $\sigma(\nabla h) = \frac{1}{\pi}\Big(L(\pi p(\Delta)) + L(\pi p(\nabla)) + L(\pi(p(\Diamond))) \Big)$

$L(z) = -\int_0^z \log|2\sin t|\, dt$

<u>This</u> has a proper formal meaning for LLN, but not for CLT

If we simply consider measure on height functions

$$\rho(\tilde{h}) \overset{?}{=} \exp\left(L^2 \iint \sigma(\nabla \tilde{h})\, dx\, dy\right)$$

Then maximum of $\iint$ will give a typical height function. ~~Linearizing near~~ Taylor expand the functional near extremum. Linear part will vanish (Euler-Lagrange). Quadratic will give $\exp\left(\iint G_2 \left(\nabla(h-\tilde{h}) \nabla(h-\tilde{h})\right)\right)$

quadratic

In appropriate coordinate system this is

$$\exp\left(-\iint \langle \nabla(h-\tilde{h}), \nabla(h-\tilde{h})\rangle\right)$$

$$= \langle h-\tilde{h}, \Delta(h-\tilde{h})\rangle$$

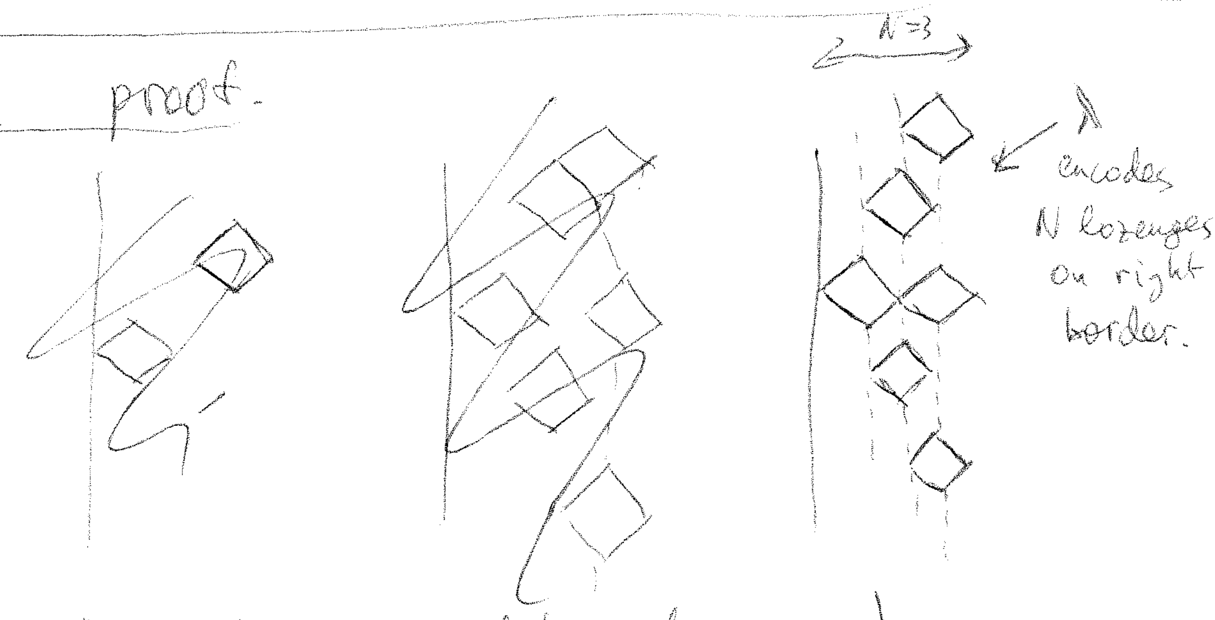
$\Rightarrow$ We see quadratic structure (Gaussians).

Covariance $\longrightarrow$ inverse matrix / operator $\Rightarrow$ Green function (General formula for Gaussians).

$\uparrow$ __Never__ made rigorous

The actual proof.

Trapezoid



The measure is uniform, conditionally on λ

Proposition (*): The S.G.F. of t lozenges at the section $x = t$ is

$$\frac{S_\lambda(x_1, \dots x_t, 1^{N-t})}{S_\lambda(1^N)}.$$

# of tilings of a trapezoid

Proof: This is branching rule for Schur polynomials (i.e. what happens when we plug $x_N = 1$). ∎

Corollary: We can apply the main theorem on S.G.F. to get gaussianity + covariance along a single section: Slight extension → joint limit.

Remains to identity double contour integral in the answer of [But.-Gor] with double integral of GFF. (takes efforts, but doable).

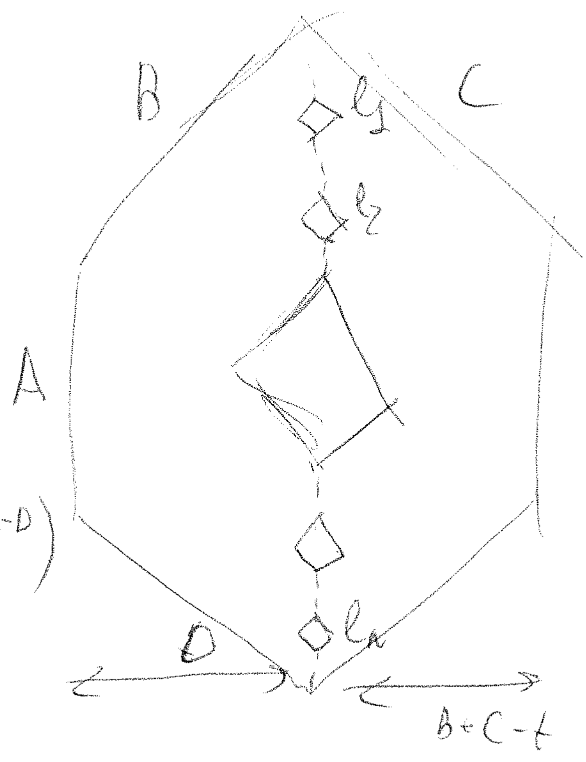
Holey hexagon

Condition on horizontal lozenges along the axis.

$P(l_1, \dots l_N) \sim$

$\sim S_{\lambda(l)}(1^D) S_{\tilde{\lambda}(l)}(1^{B+C-D})$

deterministic procedure applied to $\vec{l}$



Use $S_\lambda(1, \dots 1) = \lim_{q \to 1} S_\lambda(1, q, \dots q^{M-1}) =$

$$= \prod_{i<j} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

Thus $P(\ell_1, \dots \ell_n)$ is $\beta = 2$ log-gas!

[virtual particles in λ, $\tilde{\lambda}$ will create potential]

See handout for the exact formula.

Theorem of [Bor - Gor - Gui] gives LLN, CLT.

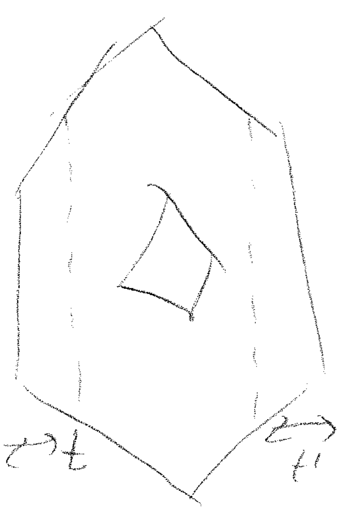
Consider multivariate S.G.F. $\sum P(\vec{\ell}) \frac{S_{\lambda(\ell)}(x_1, \dots x_D) \, S_{\tilde{\lambda}(\ell)}(y_1, \dots y_{B+C-D})}{S_{\lambda(\ell)}(1^D) \, S_{\tilde{\lambda}(\ell)}(1^{B+C-D})}$

$$S_\rho(x_1, \dots x_D; y_1, \dots y_{B+C-D}).$$

Slight extension of main theorem of [Buf-Gor] $\Rightarrow$

Partial derivatives of this S.G.F.

An analogue of proposition (*)

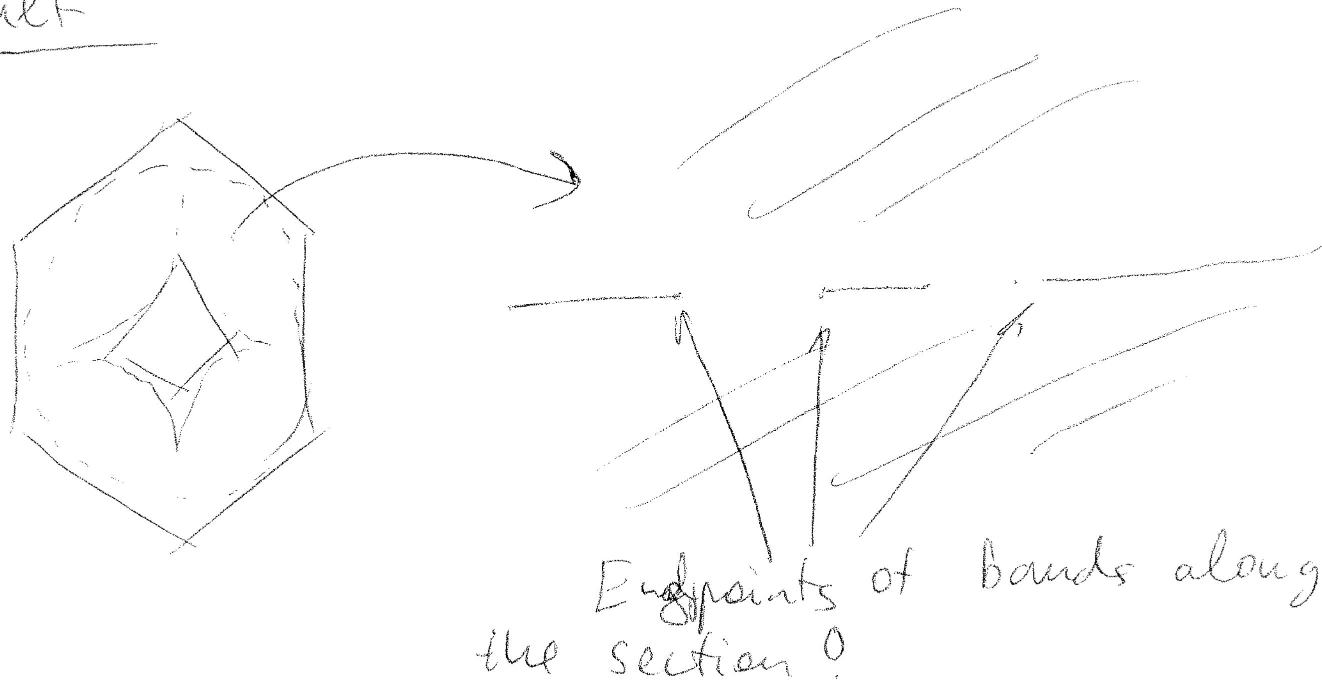


Joint S.G.F. along two such sections is $S_\rho(x_1, \dots x_t, 1^{D-t}; y_1, \dots y_{t'}, 1^{B+C-D-t'})$

So we know it!

Again by a slight extension of the main theorem, we get CLT and covariance everywhere. Remains to identify.

Result



And we get a pullback of GFF in this domain with annulus topology.