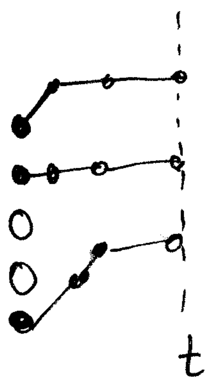
New topic: Schur generating functions. NYU Lecture 3 (1)

We still want LLN / CLT for random $\lambda_1 \geq \dots \geq \lambda_N$

Last 2 lectures: distribution is somewhat known and is given by log-gas (discrete).

<u>Now</u>: When no simple formulas for the distribution itself exist, what can we do?

<u>Two examples</u>



Distribution at time t of non-int r.w. with <u>arbitrary</u> initial condition.

T_λ = irrep of $\mathcal{U}(N)$, $\lambda_1 \geq \dots \geq \lambda_N$ (who knows? $\mathcal{U}(1)$ at least?)

$$T_\lambda \otimes T_\mu = \bigoplus C_{\lambda\mu}^{\nu} T_\nu$$

Littlewood-Richardson coefficients — hard combinatorial object ($\overset{?}{=}0$ Horn, Knutson-Tao, Klyachko).

<u>Biane</u> (90's): treat as prob. measure?

$$P(\nu) = \frac{C_\nu \cdot \dim T_\nu}{\dim T_\lambda \cdot \dim T_\mu}$$

Analogy in continuum $\longrightarrow$ random matrices.

Dyson Brownian Motion with arbitrary initial conditions

$=$ eigenvalues of $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} + GUE$?

eigenvalues of $A+B$? where A, B ~~has~~ independent <u>uniform</u> eigenvectors and <u>fixed</u> eigenvalues $\{a_i\}$, $\{b_i\}$

RM problems are well-known and solved.
One ~~efficient~~ method $\longrightarrow$ moments!

C: e.v. $c_1, \dots c_N$

$$\mathbb{E}\left(\operatorname{Tr} C^k\right)^m = \mathbb{E}\left(\Sigma (c_i)^k\right)^m$$

compute through matrix elements

gives LLN/CLT for polynomial statistics.

Discrete case remained open until recently.
No RM $\Rightarrow$ How to compute moments?

Approach of ~~[G. - Buf.]~~ [Bufetov - G.]

Alexey!

Schur polynomial (who knows?)

$$S_\lambda(x_1, \dots x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

Homogeneous Laurent symmetric polynomial of degree $|\lambda| = \lambda_1 + \dots + \lambda_N$.

<u>Def.</u> Schur generating function of probability measure $P(\lambda)$ on $\lambda_1 \geq \dots \geq \lambda_N$ is

$$S_P(x_1, \dots x_N) = \sum_\lambda P(\lambda) \frac{S_\lambda(x_1, \dots x_N)}{S_\lambda(1, \dots 1)}$$

Always converges on torus $|x_1| = \dots = |x_N| = 1$,
we will <u>assume</u> convergence on $1-\varepsilon < |x_i| < 1+\varepsilon$ $\longrightarrow$
$\longrightarrow$ tail behavior (trivially true in most natural examples)

Examples:

1) $N=1$ $\qquad \sum P(k)\, x^k$ – usual moment generating function

2) Not int. random walks

$$S_{P_t} = \frac{S_{\lambda^0}(x_1, \ldots x_N)}{S_{\lambda^0}(1, \ldots 1)} \prod_{i=1}^{N} \left(1-p+p\cdot x_i\right)^t$$

at time t $\qquad$ initial condition

„Proof" $\quad t=0 \longrightarrow$ immediate.

$$S_\lambda \cdot \prod_{i=1}^{N} \left(1-p+p\cdot x_i\right) =$$

$$= \sum_{\mu:\ \mu_i=\lambda_i+\{0/1\}} p^{|\mu|-|\lambda|} (1-p)^{N-(|\mu|-|\lambda|)} \qquad S_\mu$$

These are independendet jumps.

Conjugation by $S_\lambda(1, \ldots 1) \to$ transitional probabilities for <u>non-intersecting</u> walks.

3) Tensor products

$$S_{P_\nu} = \frac{S_\lambda(x_1, \ldots x_N)\, S_\mu(x_1, \ldots x_N)}{S_\lambda(1, \ldots 1)\, S_\mu(1, \ldots 1)}$$

Schur function = character of T_λ =

$= \text{Trace } T_\lambda(\mathcal{U})$

unitary matrix with e.v. $= x_1, \ldots x_N$.

Character of tensor product = product of characters

4) See handout?

Lozenge tilings: Section $\dfrac{S_\lambda(x_1, .. x_k, 1^{N-k})}{S_\lambda(1^N)}$

Domino tilings: Section $\dfrac{S_\lambda(x_1 .. x_k, 1^{N-k})}{S_\lambda(1^N)} \cdot \left(\dfrac{1+x_i}{2}\right)^k$

Main Theorem (Bufetov - G.).

Take a sequence of measures P_N, on $\lambda_1 \geq .. \geq \lambda_N$, $N = 1, 2, ..$

The LLN $\quad \dfrac{1}{N} \sum_{i=1}^{N} \left(\dfrac{\lambda_i + N - i}{N}\right)^k \longrightarrow p(k) \quad (\forall k)$

[polynomial linear statistics]

and CLT: $\mathbb{E}\left(\sum_{i=1}^{N} \left[\left(\dfrac{\lambda_i+N-i}{N}\right)^k - \mathbb{E}\left(\dfrac{\lambda_i+N-i}{N}\right)^k \right] \cdot \right.$

covariance → $\left. \cdot \sum_{i=1}^{N} \left[\left(\dfrac{\lambda_i+N-i}{N}\right)^\ell - \mathbb{E}\left(\dfrac{\lambda_i+N-i}{N}\right)^\ell \right] \right) \to \operatorname{cov}(k, \ell) \quad (\forall k, \ell)$

higher moments → $\sum \left(\dfrac{\lambda_i+N-i}{N}\right)^k - \mathbb{E}\left(\dfrac{\lambda_i+N-i}{N}\right)^k$ asymptotically (jointly) gaussian.

Hold if and only if

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1) $\dfrac{1}{N}\left(\dfrac{\partial}{\partial x_1}\right)^k \ln S_{P_N} \Big|_{x_1 = .. = x_N = 1} \longrightarrow C_k \quad (k = 0?)$

can be any i?

2) $\left(\dfrac{\partial}{\partial x_1}\right)^k \left(\dfrac{\partial}{\partial x_2}\right)^\ell \ln S_{P_N} \Big|_{x_1 = .. = x_N = 1} \longrightarrow d_{k,\ell}$

can be i and j

3) $\prod_{a=1}^{k} \left(\dfrac{\partial}{\partial x_{i_a}}\right) \ln S_{P_N} \Big|_{x_1 = .. = x_N = 1} \to 0$ when $|\{i_a\}| > 2$

Where

$$p(k) = [z^{-1}] \frac{1}{(k+1)(z+1)} \left((1+z) \left(\frac{1}{z} + \sum_{a=1}^{\infty} \frac{c_a z^{a-1}}{(a-1)!} \right) \right)^{k+1}$$

$$\mathrm{cov}(k, l) = [z^{-1} w^{-1}] \left((1+z) \left(\frac{1}{z} + \sum \frac{c_a z^{a-1}}{(a-1)!} \right) \right)^{k} \cdot$$

$$\cdot \left((1+w) \left(\frac{1}{w} + \sum_{a=1}^{\infty} \frac{c_a w^{a-1}}{(a-1)!} \right) \right)^{l} \cdot \left(\underbrace{\left(\sum_{a=0}^{\infty} \frac{z^a}{w^{1+a}} \right)^2}_{= 1/(z-w)^2} + \sum_{a,b=1}^{\infty} \frac{d_{a,b}}{(a-1)!(b-1)!} z^{a-1} w^{b-1} \right)$$

$\boxed{\text{Break ???}}$

1-d analogy: $\xi_1, \xi_2, \ldots$ Char. functions.

$$K_m(\xi) = \left(\frac{\partial}{\partial x} \right)^m \ln \mathbb{E} \exp(i x \xi) \Big|_{x=0} \cdot \frac{1}{i^m}$$

[under technical conditions]

1) $K_1(\xi_n) \xrightarrow{n\to\infty} c, \quad K_2(\xi_n) \xrightarrow{n\to\infty} 0 \iff \xi_n \xrightarrow{n\to\infty} c$

2) $K_2(\xi_n) \to \sigma^2, \quad K_m(\xi_n) \xrightarrow{n\to\infty} 0, \quad m > 2$

$$\Updownarrow$$

$$\xi_n - \mathbb{E}\xi_n \xrightarrow{n\to\infty} N(0, \sigma^2).$$

Same „ln", same orders of derivatives, same gaussianity

But only 1 variable!

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why Schur gen. functions?

$\mathbb{E} \exp(i x \xi) \quad (\xi \in \mathbb{Z})$

multiplicative characters of unit circle are parameterized by integers.

$\mathbb{E} \frac{S_\lambda(x_1, \ldots x_N)}{S_\lambda(1, \ldots 1)}$

characters of $U(N)$ parameterized by $\lambda_1 \geq \ldots \geq \lambda_N$

$\boxed{\text{So we extended harmonic analysis on unit circle to unitary groups!}}$

what's coming next.

<u>Today</u>: application of LLN $\to$ free probability

<u>4th lecture</u>: elements of proof

<u>5th lecture</u>: application of CLT $\to$ gaussian free field.

We now return to tensor products problem

$$T_\lambda \otimes T_\mu = \bigoplus c_\nu T_\nu$$ discrete version of $A + B = C$

$$P(\nu) = \frac{c_\nu \dim T_\nu}{\dim T_\lambda \dim T_\mu}$$

$N \to \infty$

„Free convolution"

<u>Theorem</u> (Bufetov-Gorin): Assume that $\lambda = \lambda(N)$, $\mu = \mu(N)$ is such that

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\lambda_i + N - i}{N}} \to \rho \qquad \frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\mu_i + N - i}{N}} \to \rho'$$

Then $$\frac{1}{N} \sum_{i=1}^{N} \delta_{\frac{\nu_i + N - i}{N}} \xrightarrow{\text{weakly, in prob.}} \rho \boxtimes \rho'$$

random   deterministic.

Moreover, set $\cancel{G_\mu(z) = \int}$ $\quad G_\rho(z) = \int \frac{\rho(x)}{z - x} dx$

$$R^q_\rho(z) = \left(G_\rho(z)\right)^{(-1)} - \frac{1}{1 - e^{-z}} = \text{power series in } z.$$

„Quantized R-transform"

Then $$R^q_\rho(z) + R^q_{\rho'}(z) = R^q_{\rho \boxtimes \rho'}(z)$$, i.e.

this operation on measures is <u>linearized</u> by R^ρ — just as log of Fourier transform linearizes usual convolution

Comparison

in RM setting a similar theorem is due to Voiculescu. Then $R^{\lambda}(z)$ is replaced by $R(z) = \cancel{g}\left(G_p(z)\right)^{(-1)} - \frac{1}{z}$,

operation — „free convolution" — ~~spectrum of~~ the sum of two free random variables (= spectrum of sum of 2 operators in ∞ dim space).

In our setting $\lambda_i \sim N$ (discrete).
If they grow faster than N, then tensor products degenerate to sums in „semiclassical limit"
(Group $\mathcal{U}(N)$ $\rightsquigarrow$ its Lie algebra = skew-sym. complex matrices = Hermitian complex matrices)

and one gets $R(z)$ directly (Biane; Collins-Sniady)

Proof of theorem

S.G.F.

$$S_{\sigma_\lambda} = \frac{s_\lambda(x_1, \dots x_N)}{s_\lambda(1^N)}$$

$$S_{\sigma_\mu} = \frac{s_\mu(x_1, \dots x_N)}{s_\mu(1^N)}$$

$$S_\nu = \frac{s_\lambda(x_1, \dots x_N)\, s_\mu(x_1, \dots x_N)}{s_\lambda(1^N) \quad s_\mu(1^N)}.$$

So the logarithms are added.

λ, μ satisfy LLN + CLT with 0 covariance.

$\Rightarrow$ conditions of theorem hold, i.e.

$\ln S_{\delta_\lambda}$, $\ln S_{\delta_\mu}$ satisfy 1), 2) 3) $\Rightarrow$ so is S_ν $\Rightarrow$

$\Rightarrow$ LLN + CLT for S_ν holds.

Linearization: By definition, c_k in 1) are added, i.e.

$$\frac{\partial}{\partial x} \lim_{N\to\infty} \frac{1}{N} \ln\left(S(x+1, 1^{N-1})\right) = \sum_{a=1}^{\infty} \frac{c_a x^{a-1}}{(a-1)!}$$ linearizes.

It remains to use $p(k) \longleftrightarrow c_a$ relation and simplify the result.

(version of Lagrange inversion).

<u>Conclusion</u>: You do not really need to know <u>anything</u> about Schur functions to use our theorem!