These lectures are about random N-particle discrete configurations $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \in \mathbb{Z}$, as $N \to \infty$.

They come from several sources:

1) 2d statistical mechanics
2) Interacting particle systems
3) Asymptotic Representation Theory

1) → Handout

2) TASEP and relatives (one of them will be our running example)

3) Decomposition of reps of large groups (e.g. tensor products) into irreducible components

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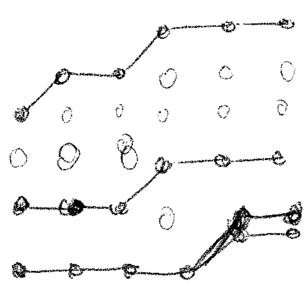
Although the methods will be quite general, but let us fix a running example to keep the discussion precise.

N independent random walks with arbitrary initial conditions, jump probability p and conditioned to have no collisions

N = 3.



Discrete analogue of $\beta=2$ Dyson Brownian Motion

What's new in these lectures?

We develop robust methods, which lead to universal results (e.g. you do not care much about the initial condition in non-intersecting paths).

This is different from the previous results of Integrable probability, which were destroyed by small perturbations.

Our focus: macroscopic behavior / linear statistics

$$\sum_{i=1}^{N} f\left(\frac{\lambda_i}{N}\right) \xrightarrow{N\to\infty} ?$$

f will be smooth

(f-indicator $\to$ particle counts)

LLN: $\frac{1}{N}\cdot(\;) \to$ const.

CLT: Gaussian fluctuations

main developments

Two approaches $\leftrightarrow$ two characterizations of Gaussian Law.

$$\xi \sim N(0,1)$$

Characterizing equation

Moments

For smooth f

$$\mathbb{E} f'(\xi) = \mathbb{E}\, \xi f(\xi)$$

integration by parts

$\xleftrightarrow{f(x)=x^n}$

(can work together!)

$$\mathbb{E}\,\xi^n = \begin{cases} 0, & n \text{ is odd} \\ (n-1)!!, & n \text{ is even} \end{cases}$$

# of perfect matchings of $\{1,..n\}$

We will develop „approximate ~~[illegible]~~ Moments formulas"

Discrete loop (Nekrasov) equations [Borodin-Gorin-Guionnet], + Borot

First 2 lectures

through Schur generating functions [Bufetov-Gorin]

Setup for <u>Nekrasov</u> <u>equations</u> (simplest?).

$\lambda_1 \geq \ldots \geq \lambda_N \in \mathbb{Z}$ $\qquad \theta > 0$

$l_i = \lambda_i - \theta i \qquad a \geq l_1 > \ldots > l_N \geq b$

$a - l_1 \in \mathbb{Z}_{\geq 0} \qquad l_N - b \in \mathbb{Z}_{\geq 0}$

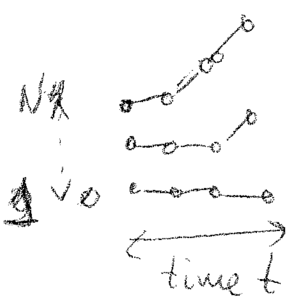
$$\mathbb{P}(\lambda_1, \ldots \lambda_N) = \frac{1}{Z} \prod_{i<j} \left[(l_i - l_j) \frac{\Gamma(l_i - l_j + \theta)}{\Gamma(l_i - l_j + 1 - \theta)} \right] \prod_{i=1}^{N} w(l_i)$$

w - analytic, > 0 on (a,b), $w(a) = w(b) = 0$.

<u>Examples:</u> $\theta = 1 \longrightarrow \prod_{i<j} (l_i - l_j)^2$

<u>l_i - integers?</u>

densely packed initial condition



$$\frac{1}{Z} \prod_{i<j} (l_i - l_j)^2 \prod_{i=1}^{N} \binom{N+t-1}{l_i - 1} p^{l_i - 1} (1-p)^{N+t-l_i}$$

on $\{1, \ldots N+t\}$

$$\frac{(N+t-1)!}{\Gamma(l_i)\,\Gamma(N+t-l_i+1)}$$

$a = 0, \quad b = N + t + 1.$

~~For other~~ Why? Karlin-McGregor / Gessel-Viennot formula for # of non-intersecting paths <u>(interest?)</u>

Other $\theta = 1$ examples $\longrightarrow$ see handout.

General θ

1) Fix θ, $l_j, l_i \to \infty$ $\quad (l_j - l_i)\dfrac{\Gamma(l_i - l_j + \theta)}{\Gamma(l_i - l_j + (1-\theta))} \approx$

$$\approx (l_j - l_i)^{2\theta}$$

This is a discrete analogue of general β log-gases in random matrix theory ($\beta = 2\theta$ = dimension of base field).

2) Same interaction factor shows up in evaluation formulas for Jack symmetric functions (Rep. theory (zonal polynomials) + integrable systems (Calogero-Sutherland,))

Important: $l_i = \lambda_i - \theta i$ - very special "lattice"!

$$l_{i+1} - l_i \in \{\theta, \theta+1, \theta+2, \ldots\}$$

Main technical tool "Nekrasov equation":

Assume $\dfrac{w(x)}{w(x-1)} = \dfrac{\varphi^+(x)}{\varphi^-(x)}$ ← analytic in a neighborhood M of $[a,b]$.

[$w(a) = w(b) = 0 \Rightarrow$ ~~$\varphi^-(a) = 0$~~ $\varphi^-(a+1) = \varphi^+(b) = 0$]

Define $R_N(z) = \varphi^-(z)\, \mathbb{E} \prod_{i=1}^{N} \left(1 - \dfrac{\theta}{z - l_i}\right) +$

$$\varphi^+(z)\, \mathbb{E} \prod_{i=1}^{N} \left(1 + \dfrac{\theta}{z - l_i - 1}\right)$$

Then $R_N(z)$ is analytic in M, i.e. it has no poles

This is an equation

(5)

Guessing right form is hard $\to$ originates in similar expressions in the work of Nekrasov and collaborators. Uses <u>all</u> features of the definition

$\boxed{\text{Proof}}$ We compute the residue at a possible pole $x = n - m\cdot\theta$

Fix $i \in \{1, \dots N\}$ $\mathbb{E}$ = sum over configulations Those which have $l_i = x$ and $l_i = x-1$ contribute to the pole

1st term $\vec{l}_i = x$ 2nd term

Take a configuration $\vec{l}$ and let $\vec{l}_-$ be obtained by $l_i \to l_i - 1$. Then

$$\frac{P(\vec{l}_-)}{P(\vec{l})} = \frac{w(x-1)}{w(x)} \cdot \prod_{j\neq i} \frac{l_i - 1 - l_j}{l_i - l_j} \cdot \frac{l_i - l_j - \theta}{l_i - l_j + \theta - 1}$$

$= \frac{\varphi^-(x)}{\varphi^+(x)}$ Two cases $j<i$, $j>i$ lead to same formula!

$$\theta P(\vec{l}_-)\varphi^+(x)\cdot\prod_{j\neq i}\left(1 + \frac{\theta}{l_i - l_j - 1}\right) =$$

$$= \theta P(\vec{l})\, \varphi^-(x)\cdot\prod_{j\neq i}\left(1 - \frac{\theta}{l_i - l_j}\right)$$

So residue at x from $\vec{l}$ + from $\vec{l}_-$ $= 0$

$\underline{\qquad} \; || \; \underline{\qquad}$

Important: $\vec{\ell}_-$ might fail to be in our state space (e.g. if $\ell_i - \ell_{i-1} = \theta$)

However, in this case $P(\vec{\ell}_-) = 0$ and r.h.s $= 0$

$\varphi^-(x) = 0$ or $\left(1 - \frac{\theta}{\ell_i - \ell_j}\right) = 0$

Similarly, if $\vec{\ell}_-$ is in the state space, but $\vec{\ell}$ is not.

That's where all the features of the definition of ~~~~ discrete log-gas play a role ∎

How to use it? First application: explicit LLN

Assume $w(x) = \exp\left(-N V\left(\frac{x}{N}\right) + \varepsilon_N\right)$, where

V is continuous in $\left[\frac{a}{N}, \frac{b}{N}\right]$, and $|V'(\xi)| \le C\left(1 + \ln\left|\xi - \frac{a}{N}\right| + \ln\left|\xi - \frac{b}{N}\right|\right)$,

converge

and $|\varepsilon_N| \le C \ln(N)$.

Theorem: Then $\mu_N := \frac{1}{N} \sum \delta_{\ell_i/N} \overset{N\to\infty}{\approx} \mu(x)dx$,

where $\mu(x)\,dx$ is the ~~minimizer~~ maximizer of

$$I[\nu] = \theta \iint \ln|x-y|\, \nu(x)\nu(y)\,dx\,dy - \int V(x)\,\nu(x)\,dx$$

on probability measures on $\left[\frac{a}{N}, \frac{b}{N}\right]$, $0 \le \nu(x) \le \theta^{-1}$

(That's because $\ell_{i+1} - \ell_i \ge \theta$)

in the form $\forall \varepsilon > 0$, Lipshitz in a neighborhood of $\frac{a}{N}, \frac{b}{N}$ function f

$$N^{\frac{1}{2}-\varepsilon} \left| \int f(x)\,\mu_N(dx) - \int f(x)\,\mu(x)\,dx \right| \longrightarrow 0$$

in probability and in the sense of moments

The proof is standard, as in continuous case. I will not present it, unless the audience is really interested. (Nothing to do with Nekrasov equations)

Given the theorem, what happens with Nekrasov equation?

Assumptions: $\xi = Nz$

$$\varphi_N^-(Nz) \to \varphi^-(z)$$
$$\varphi_N^+(Nz) \to \varphi^+(z)$$

Proposition:

$$R_N(Nz) = \varphi_N^-(Nz)\,\mathbb{E}\prod\left(1-\frac{\theta}{Nz-\ell_i}\right) + \varphi_N^+(Nz)\,\mathbb{E}\prod\left(1+\frac{\theta}{Nz-\ell_i-1}\right)$$

$\xrightarrow[\text{uniformly}]{N\to\infty}$ $R(z) = \varphi^-(z)\exp(-\theta G(z)) + \varphi^+(z)\exp(\theta G(z))$,

where $G(z) = \int \frac{\mu(x)}{z-x}\,dx$.

$R(z)$ is analytic where $\varphi^\pm$ are

Proof.

$$\prod\left(1-\frac{\theta}{Nz-\ell_i}\right) = \exp\left(\sum_{i=1}^N \ln\left(1-\frac{1}{N}\frac{\theta}{z-\ell_i/N}\right)\right) =$$

$$= \exp\left(-\theta\sum_{i=1}^N \frac{1}{z-\ell_i/N} + \frac{1}{N^2}\sum\frac{1}{(z-\frac{\ell_i}{N})^2}\cdot O(1)\right) \longrightarrow$$

for z bounded away from $[\frac{a}{N}, \frac{b}{N}]$

$\xrightarrow{\text{by Theorem}} \exp(-\theta G(z))$. Similarly for the second ~~integral~~ expectation. So choose a contour γ, such that $[\frac{a}{N}, \frac{b}{N}]$ is inside at positive distance.

$R_N(Nz) \to R(z)$ uniformly on γ.

For any w inside γ

$$R_N(Nw) = \frac{1}{2\pi i} \oint_\gamma \frac{R_N(Nz)}{z-w}\,dz$$

Cauchy formula — follows from analyticity of R_N.

$N \to \infty$ leads to $R(w) = \frac{1}{2\pi i} \oint_\gamma \frac{R(z)}{z-w}\,dz$,

which shows that $R(w)$ is analytic in w and provides analytic continuation to everywhere inside γ.

Example: Non-intersecting walks

$$w(x) = p^{x-1}(1-p)^{N+t-x} \binom{N+t-1}{x-1} \quad \text{on } \{1, \dots N+t\}$$

$$\frac{w(x)}{w(x-1)} = \frac{p}{1-p} \; \frac{N+t-x}{x-1}. \qquad t = \tau N, \quad N \to \infty$$

$$\frac{p}{1-p} \; \frac{1 + \frac{t}{N} - \frac{x}{N}}{\frac{x}{N} - \frac{1}{N}} \xrightarrow[N\to\infty]{z = \frac{x}{N}} \frac{p(1+\tau - z)}{(1-p)\,z} \qquad \begin{matrix} = \varphi^+(z) \\ = \varphi^-(z) \end{matrix}$$

Therefore, the empirical measure of walks at time τN as $N \to \infty$ has LLN satisfying

$$p(1+\tau - z)\exp(G(z)) + (1-p)z\exp(-G(z)) = R(z) \text{ — analytic in } z \in \mathbb{C}$$

$$z \to \infty \quad \exp(G(z)) \approx \exp\left(\frac{1}{z} + \underline{\underline{O}}\left(\frac{1}{z^2}\right)\right) = \left(1 + \frac{1}{z} + \underline{\underline{O}}\left(\frac{1}{z^2}\right)\right).$$

Thus $|R(z)| \le C|z|$, $z \to \infty$.

Liouville theorem: then $R(z) = \alpha z + \beta$.

On the other hand, $R(z) = p(1+\tau-z)\left(1+\frac{1}{z}\right) + (1-p)z\left(1-\frac{1}{z}\right) + \underline{\underline{O}}\left(\frac{1}{z}\right) =$

$= p(1+\tau-z) + (1-p)z - p - (1-p) - \underline{\underline{O}}\left(\frac{1}{z}\right) = (1-2p)z + p(\tau+1) - 1 + \underline{\underline{O}}\left(\frac{1}{z}\right)$

So $\alpha = (1-2p)$, $\beta = p(1+\tau) - 1$

$$p(1+\tau-z)\exp G(z) + (1-p)z\exp(-G(z)) = (1-2p)z + p(1+\tau) - 1$$

$$\Updownarrow$$

$$\left(\exp(G(z))\right)^2 p(1+\tau-z) - \exp(G(z))\left((1-2p)z + p(1+\tau)-1\right) + (1-p)z = 0.$$

Solving quadratic equation, we get $\exp(G(z))$ and $G(z)$ itself.

$$\mu(x) = \frac{1}{\pi}\operatorname{Im}\lim_{\varepsilon\to 0} G(x+i\varepsilon).$$

$\nwarrow$ $\operatorname{Arg}(\exp(G(x+i\varepsilon)))$

What's next? We will prove CLT using the same ~~equa~~ Nekrasov equation.

Main difficulty: no control over analytic $R_N(\xi)$ $\Rightarrow$ need to get rid of it somehow.