General beta random matrix theory

(at MATRIX Institute)

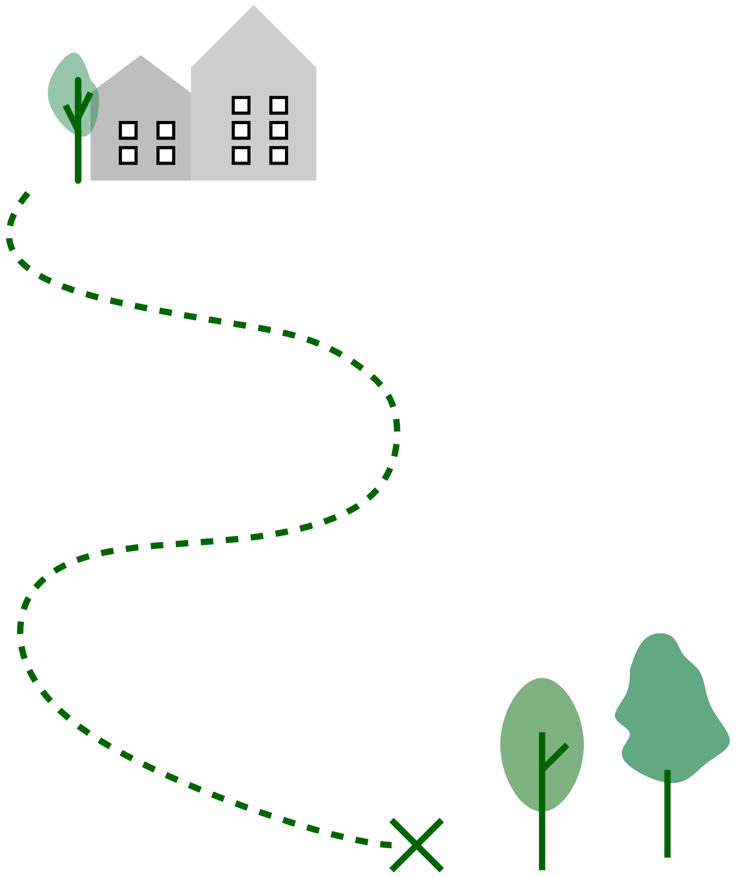
Vadim Gorin

University of Wisconsin — Madison
and
Institute for Information Transmission Problems of Russian Academy of Sciences

Lecture 2

June 2021

Roadmap



- What are general β random matrices?
- Lecture 1: corners of β random matrices.
- Problem set 1.
- Lecture 2: sums of β random matrices.
- Problem set 2.
- Lecture 3: questions and discussion of problem sets.

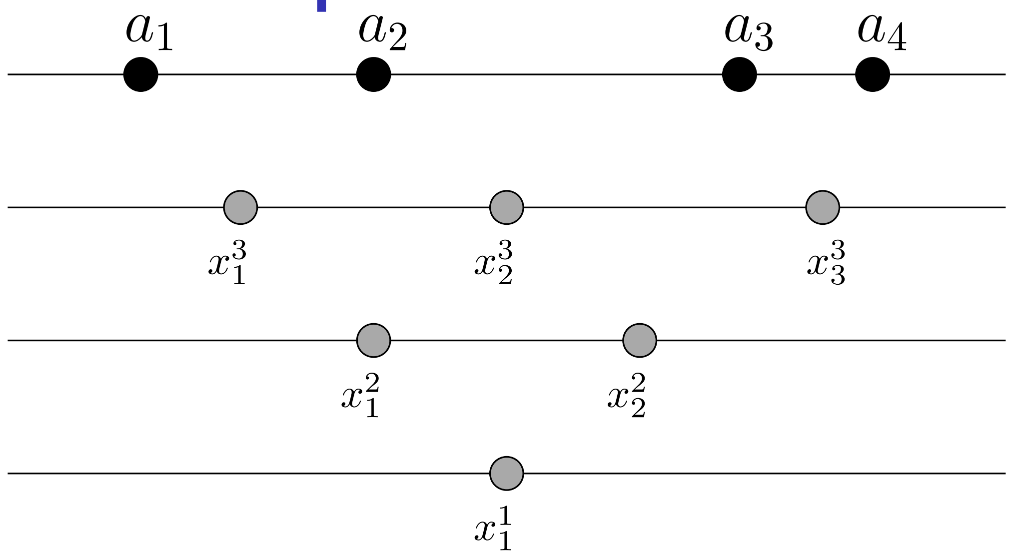
[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

This is NOT a research talk about brand new results.
Instead we explore **basic structures and definitions.**

(See "Lattice Paths, Combinatorics and Interactions" in 2 weeks).

Recap: β–corners process



Fix $\beta > 0$
$N = 1, 2, \ldots$
$a_1, \ldots, a_N \in \mathbb{R}$

Definition. Eigenvalues of corners of $N \times N$ **random β-matrix** with uniformly random eigenvectors and fixed eigenvalues $(a_i)_{i=1}^N$ are a triangular array $(x_i^k)_{1 \le i \le N}$ satisfying

$$x_{i+1}^k \le x_i^k \le x_{i+1}^{k+1}, \qquad (x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N),$$

with distribution of density

$$\left[\prod_{k=1}^{N} \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k}\right] \cdot \prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

Next question: What is the sum of random β-matrices?

Toy question: sum of independent random variables

Take independent random variables a and b

How do you think about the law of their sum $c = a+b$?

Fourier point of view: through characteristic functions

Random variable a $\Longleftrightarrow$ Function $\mathbb{E}\, e^{ita}$ of $t \in \mathbb{R}$

[The distribution of a is reconstucted by inverse Fourier tr.]

Theorem: The distribution of c is uniquely determined by

$$\mathbb{E}\, e^{itc} = \underbrace{\mathbb{E}\, e^{ita} \cdot \mathbb{E}\, e^{itb}}_{\text{something we know}}, \quad t \in \mathbb{R}$$

Proof: independence implies that the identity is true for $c = a+b$, uniqueness th. for $\mathbb{E}\, e^{itc}$ implies that no other law satisfies it.

Conclusion: To compute the law of $c = a + b$, you do not need to know „what is „+" or what are independent random variables. All you need to learn is multiplication of characteristic functions.

Sum of matrices at $\beta = 1, 2, 4.$

Theorem. Random $N \times N$ self-adjoint independent matrices A, B. The law of the sum $C = A + B$ is uniquely determined by

$$\boxed{\mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(CZ)\right) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right) \cdot \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(BZ)\right)},$$

which should be valid for each self-adjoint Z.

Proof. 1) Identity is true by $\mathrm{Trace}\, CZ = \mathrm{Trace}\, AZ + \mathrm{Trace}\, BZ$ and independence

2) $\mathrm{Trace}\,(CZ) = \sum_{ij} C_{ij} Z_{ji}$. Hence, these are usual multidimensional characteristic functions in Euclidean space of all $N \times N$ self adjoint matrices.

In particular, there is a uniqueness theorem just like for $N=1$. ∎

Reduction to eigenvalues

Definition 1. A: deterministic eigenvalues $(a_1, \ldots, a_N)$ and uniformly random eigenvectors (invariant under $A \mapsto UAU^*$). Then law of $\mathrm{Trace}(AZ)$ depends only on eigenvalues $(z_i)_{i=1}^N$ of Z and we define the **multivariate Bessel function** through

$$\boxed{B_{a_1,\ldots,a_N}(\mathbf{i}z_1, \ldots, \mathbf{i}z_N;\ \beta/2) = \mathbb{E}\exp\left(\mathbf{i}\mathrm{Trace}(AZ)\right)}$$

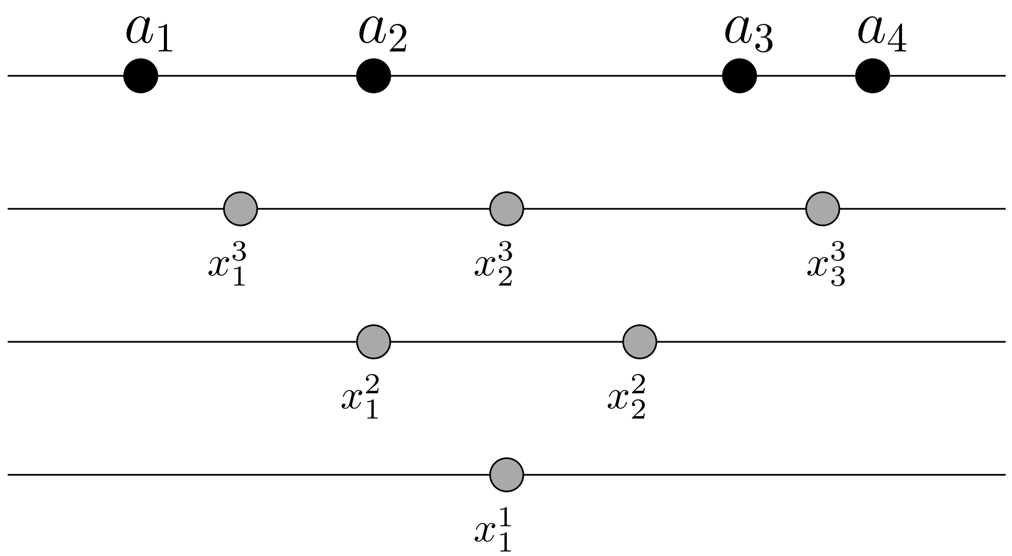
Proof. Z is assumed here to be self-adjoint or „normal" In both cases it can be diagonalized by orthogonal/unitary transform $Z = U \cdot \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{pmatrix} U^*$. Hence,

$$\mathrm{Trace}(AZ) = \mathrm{Trace}\left(AU\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{pmatrix}U^*\right) = \mathrm{Trace}\left(U^*AU\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{pmatrix}\right) =$$

$$\stackrel{d}{=} \mathrm{Trace}\left(A\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_N \end{pmatrix}\right) = \sum_{i=1}^N A_{ii} z_i.$$

Hence, $\mathbb{E}e^{i\,\mathrm{Tr}(AZ)}$ indeed depends only on the eigenvalues of Z. $\square$

Reduction to corners



Fix $\beta > 0$
$N = 1, 2, \ldots$
$a_1, \ldots, a_N \in \mathbb{R}$

Definition 2. Take β–corners process with top row $(a_i)_{i=1}^N$; $(x_i^k)_{1 \le i \le k \le N}$. The **multivariate Bessel function** is:

$$\boxed{B_{a_1,\ldots,a_N}(z_1, \ldots, z_N;\, \beta/2) = \mathbb{E} \exp \left[\sum_{k=1}^{N} z_k \left(\sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1} \right) \right]}$$

Important: This makes sense for each $\beta > 0$.

Proposition. Two definitions coincide, i.e., at $\beta = 1, 2, 4$ we have

$$\mathbb{E}\exp\left(\mathbf{i}\,\mathrm{Trace}(AZ)\right) = \mathbb{E}\exp\left[\mathbf{i}\sum_{k=1}^{N} z_k \left(\sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1}\right)\right]$$

Proof. We showed 2 slides ago that

$$\mathbb{E}\exp\left(i\,\mathrm{Trace}(AZ)\right) = \mathbb{E}\exp\left(\mathbf{i}\sum_{k=1}^{N} A_{kk} z_k\right)$$

A_{kk} = Trace of $k \times k$ corner − Trace of $(k-1)\times(k-1)$ corner

$$= \sum_{i=1}^{k} x_i^k - \sum_{i=1}^{k-1} x_i^{k-1}$$

eigenvalues.

$\blacksquare$

Eigenvalues of the sum of β random matrices

Definition. Given deterministic eigenvalues $(a_i)_{i=1}^N$ and $(b_i)_{i=1}^N$ we define (random) eigenvalues $(c_i)_{i=1}^N$ of the sum of independent β-matrices with uniformly random eigenvectors through

$$\mathbb{E} B_{c_1,\dots,c_N}(z_1,\dots,z_N;\ \beta/2) \\ = B_{a_1,\dots,a_N}(z_1,\dots,z_N;\ \beta/2)\cdot B_{b_1,\dots,b_N}(z_1,\dots,z_N;\ \beta/2)$$

- $c = a \boxplus_\beta b$ at $\beta = 1, 2, 4$ is the same old addition.
- At general $\beta > 0$ one needs to show the existence of **probability measure** defining $(c_i)_{i=1}^N$.
- It is well-defined as a generalized function (distribution), but being a measure is a known open problem.

[$\approx$ need positivity of structure constants of multiplication for **Macdonald polynomials**]

Example: β–addition at $N = 1$.

$N=1$ Bessel function:

$$B_a\left(z; \tfrac{\beta}{2}\right) = \mathbb{E}\, e^{z(\overset{= a}{x_1^1})} = e^{za}$$

$c = a+b$ is defined by

$$\mathbb{E}_c\, e^{zc} = e^{za} \cdot e^{zb} = e^{z(a+b)}$$

Hence, $c = a+b$ almost surely and this is the usual addition!

[1×1 matrix does not have nontrivial eigenvectors which complicate our life for $N>1$]

What are $B_{a_1,\ldots,a_N}(z_1,\ldots,z_N;\ \frac{\beta}{2})$?

- **Symmetric** functions in $z_1,\ldots,z_N$ | easy at $\beta=1,2,4$ through Trace (AZ)
- Limits of **Jack or Macdonald polynomials**.

$N=1:$
$$e^{az} = \lim_{m\to\infty}(1+z/m)^{\lfloor ma \rfloor}$$

- Explicit Taylor **series expansion** in Jack polynomials.

$N=1:$
$$e^{az} = 1 + az + \frac{(az)^2}{2!} + \frac{(az)^2}{3!} + \ldots$$

$$B_{a_1,\ldots,a_N}(z_1,\ldots,z_N;\ \tfrac{\beta}{2}) = \sum_\mu \frac{P_\mu(z_1,\ldots,z_N;\ \frac{\beta}{2})Q_\mu(a_1,\ldots,a_N;\ \frac{\beta}{2})}{(N\frac{\beta}{2})_\mu}$$

Jack polynomials with 2 normalizations (like in Cauchy identity)

- **Eigenfunctions** of (symmetric) Dunkl operators

$$D_i := \frac{\partial}{\partial z_i} + \frac{\beta}{2}\sum_{j:j\neq i}\frac{1}{z_i-z_j}\circ(1-s_{i,j})$$

swaps i-th and j-th variable

$$\sum_{i=1}^N (D_i)^k B_{a_1,\ldots,a_N}(z_1,\ldots,z_N;\ \tfrac{\beta}{2}) = \sum_{i=1}^N (a_i)^k B_{a_1,\ldots,a_N}(z_1,\ldots,z_N;\ \tfrac{\beta}{2})$$

[See PSET 2 for checks at $\beta=2$]

Theorem: At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ has the form: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

Proof I. Let us try to find $\beta=0$ Bessel functions through the properties of the last slide:

They should be symmetric and satisfy

$$\left(\sum_{i=1}^{N} \left(\frac{\partial}{\partial z_i}\right)^k \right) B_{a_1..a_N}(z_1,..z_N;0) = \sum a_i^k \, B_{a_1,..a_N}(z_1,..,z_N;0)$$

What are eigenfunctions of differentiations?

Exponents! Hence, here we need symmetric exponents:

$$B_{a_1..a_N}(z_1,..z_N,0) = \text{const} \cdot \sum_{\sigma\in S(N)} \prod_{i=1}^{N} e^{z_i a_{\sigma(i)}}$$

What is const? $B(0,..0)=1 \Rightarrow \text{const} = \frac{1}{N!}$

Theorem: At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ has the form: Choose a permutation $\sigma \in S(N)$ uniformly at random and set $(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

Proof II. Hence, $(C_1, \ldots C_N)$ is found from

$$\mathbb{E}_c \frac{1}{N!} \sum_{\sigma_1} \prod_{i=1}^{N} e^{z_i c_{\sigma_1(i)}} = \frac{1}{(N!)^2} \sum_{\sigma_2, \sigma_3} \prod_{i=1}^{N} e^{z_i \left(a_{\sigma_2(i)} + b_{\sigma_3(i)}\right)}$$

Renaming the permutations, we get the statement of the theorem ∎

Expected characteristic polynomial

Theorem. At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ is:
Choose a permutation $\sigma \in S(N)$ uniformly at random and set
$(c_1, \ldots, c_N) = (a_1 + b_{\sigma(1)}, \ldots, a_N + b_{\sigma(N)})$.

Corollary. At $\beta = 0$, we have

$$\mathbb{E} \prod_{i=1}^{N} (z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N} (z - a_i - b_{\sigma(i)}).$$

$\simeq$ „$\mathbb{E} \det(z - C)$"

Theorem. The last expectation identity holds **for all $\beta \in [0, +\infty]$.**

[At $\beta = \infty$, expectation sign can be removed.]

as for corners we had convergence to a deterministic operation as $\beta \to \infty$. Similarly here c_i are non-random as $\beta \to \infty$

Expected characteristic polynomial

Theorem. At $\beta = 0$ the operation $(a, b) \mapsto c = a \boxplus_0 b$ is:
Choose a permutation $\sigma \in S(N)$ uniformly at random and set
$(c_1, \dots, c_N) = (a_1 + b_{\sigma(1)}, \dots, a_N + b_{\sigma(N)})$.

Corollary. At $\beta = 0$, we have

$$\mathbb{E} \prod_{i=1}^{N} (z - c_i) = \frac{1}{N!} \sum_{\sigma \in S(N)} \prod_{i=1}^{N} (z - a_i - b_{\sigma(i)}).$$

Theorem. The last expectation identity holds **for all $\beta \in [0, +\infty]$.**
[At $\beta = \infty$, expectation sign can be removed.]

Hint on the proof. [using Taylor expansion in Jacks]

- Expectations of Jack polynomials in eigenvalues $(c_1, \dots, c_N)$.
- One-column Jacks do not depend on β:
  $P_{(1^k)}(c_1, \dots, c_N; \frac{\beta}{2}) = e_k(c_1, \dots, c_N)$. elementary symmetric functions
- There are coefficients of expected characteristic polynomial.

Another asymptotic result: free convolution

Theorem. Suppose that as $N \to \infty$

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i/N} \to \mu_a, \qquad \text{with} \qquad G_{\mu_a}(z) = \int \frac{\mu_a(dx)}{z-x},$$

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{b_i/N} \to \mu_b, \qquad \text{with} \qquad G_{\mu_b}(z) = \int \frac{\mu_b(dx)}{z-x}.$$

Then for $c = a \boxplus_\beta b$

„free convolution of μ_a and μ_b"

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{c_i/N} \to \mu_c, \qquad \text{with} \qquad G_{\mu_c}(z) = \int \frac{\mu_c(dx)}{z-x},$$

$$R_\mu(z) = (G_\mu(z))^{(-1)} - \frac{1}{z}, \qquad \boxed{R_{\mu_c}(z) = R_{\mu_a}(z) + R_{\mu_b}(z)}.$$

[Written down at $\beta=1,2,4,\infty$]

Holds for each $\beta > 0$, but **not** for $\beta = 0$. [Ordinary convolution instead at $\beta=0$]

[Come back to my talk in two weeks for the critical $\beta N \to \gamma$ regime.]

End of Lecture 2. Don't forget about Problem set 2.

End of Lecture 2.

Don't forget about Problem set 2.