General beta random matrix theory
(at MATRIX Institute)

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Lecture 1
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Roadmap
• What are general $\beta$ random matrices?
• Lecture 1: corners of $\beta$ random matrices.
• Problem set 1.
• Lecture 2: sums of $\beta$ random matrices.
• Problem set 2.
• Lecture 3: questions and discussion of problem sets.

[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

This is NOT a research talk about brand new results. Instead we explore basic structures and definitions.
(See “Lattice Paths, Combinatorics and Interactions” in 2 weeks).
Random matrix theory

The study of random large matrices and their eigenvalues.

Origins:

• Representation theory of the classical groups since 1920s.
  [Groups of matrices come with normalized measures.]
• Multidimensional statistics since 1930s.
  [Data is random and is naturally organized in 2-dimensional arrays.]
• Theoretical physics since 1950s.
  [Energy levels in heavy nuclei modelled by eigenvalues.]
• Number theory since 1970s.
  [Zeros of Riemann zeta-function modelled by eigenvalues.]
• Reemphasized in modern applied and statistical problems.
  [“Big data” revolution.]

The central and the most basic random matrix object is the Gaussian Orthogonal/Unitary/Symplectic Ensemble.
Gaussian $\beta$ ensembles

$N \times N$ matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

$$M = \frac{X + X^*}{2} = \begin{pmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} \\ \vdots & \ddots \end{pmatrix}$$

The density of eigenvalues $x_1 < x_2 < \cdots < x_N$:

$$\sim \prod_{1 \leq i < j \leq N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

$\beta = 1, 2, 4$ is the dimension of the base (skew-) field.

After today’s lecture and pset you should be able to prove it!
Gaussian $\beta$ ensembles

$$\prod_{1 \leq i < j \leq N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

First correlation function for $N = 3$:

$$\frac{1}{3} \mathbb{E} \left[ \delta_{x_1} + \delta_{x_2} + \delta_{x_3} \right]$$

Five meaningful values ask for a unified treatment of $\beta \in [0, +\infty]$

This is the topic of general $\beta$ random matrix theory.
Tasks of $\beta$ random matrix theory

- **Asymptotic questions:** E.g., $N \to \infty$ behavior of density
  
  \[
  \prod_{1 \leq i < j \leq N} (x_j - x_i)^{\beta} \prod_{i=1}^{N} V(x_i). 
  \]

  with fixed $\beta > 0$, or $\beta \to 0$, or $\beta \to \infty$.

- **Algebraic questions:**
  How do we add and multiply general $\beta$ random matrices?

  ![Diagram](diagram.png)

  **Disclaimer:** There is no field of dimension $\beta$. 

  \[
  OPEN \quad This \ course
  \]
Algebra: Rank 1 operations as a building block.

\[ C = A + B \]  \hspace{0.5in} \text{(Additive version)}

- Rank 1: a single non-zero eigenvalue

Weyl's inequalities relate eigenvalues of $A$ and $C$. The distributions are also the simplest.

\[ C = P A P \]  \hspace{0.5in} \text{(Multiplicative version)}

- Rank $(N-1)$ projector
- Arbitrary $N \times N$

If $P$ projects on the first $(N-1)$ coordinate vectors

\[ A = (C I) \] — we cut corners

Additive and multiplicative math are similar. We build our theorems on the latter
The key computation

$N \times N$ matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables with real parts $\mathcal{N}(0, \frac{2}{\beta})$. $M = \frac{X+X^*}{2}$.

$$
\begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
$$

Eigenvalues:

- $(\lambda_i)^N_{i=1} \rightarrow N \times N$
- $(\mu_i)^{N-1}_{i=1} \rightarrow (N-1) \times (N-1)$

Theorem. Conditional distributions are:

1. $(\mu_i)$ given $(\lambda_i)$ solve $\sum_{i=1}^{N} \frac{\xi_i}{z - \lambda_i} = 0$.

2. $(\lambda_i)$ given $(\mu_i)$ solve $\sum_{i=1}^{N-1} \frac{\xi'_i}{z - \mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$.

$\xi_i$ and $\xi'_i$ are i.i.d. $\frac{4}{\beta} \chi^2_{\beta}$ random variables, that is $\sum_{j=1}^{\beta} \mathcal{N}_j^2(0, \frac{4}{\beta})$.

Important: This is a basis of extension to all $\beta \in [0, +\infty]$. 
Proof that \((\lambda_i)\) given \((\mu_i)\) solve

\[
\sum_{i=1}^{N-1} \frac{\xi_i}{z - \mu_i} = z + \mathcal{N}(0, \frac{2}{\beta}) : \quad I
\]

\[
M = \begin{pmatrix}
A & M_{MN} \\
M_{NM} & M_{NN}
\end{pmatrix}
\]

\text{(We deal with } \beta=2 \text{ case)}

\[\beta=1, 4 \text{ is the same}\]

\(A\) is self-adjoint \(\Rightarrow\) \exists unitary matrix \(U \in O(N-1)\)

such that \(U A U^* = \begin{pmatrix} m_{ij} & 0 \\ 0 & 0 \end{pmatrix}\)

Set \(\tilde{U} = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix}\) and replace \(M \rightarrow \tilde{U} M \tilde{U}^*\)
Proof that \((\lambda_i)\) given \((\mu_i)\) solve

\[
\sum_{i=1}^{N-1} \frac{\xi'_i}{z - \mu_i} = z + \mathcal{N}(0, \frac{2}{\beta}) : \quad II
\]

we get

\[
M' = \begin{pmatrix}
\mu_1 & 0 & p_2 \\
0 & \mu_{N-1} & \rho_{N-1} \\
\rho_1 & \rho_{N-1} & q
\end{pmatrix}
\]

Claim: Given \(\mu_1, \ldots, \mu_{N-1}, \gamma_1, \ldots, \gamma_{N-1}\) are i.i.d. complex Gaussians, and \(q\) is a real \(\mathcal{N}(0, \frac{2}{\beta})\)

Indeed, \((\rho_1, \ldots, \rho_{N-1})\) obtained from last column of \(M\) by rotation with independent unitary matrix, which preserves being i.i.d. Gaussians.

\(q\) was not changed in \(M \rightarrow UMU^*\) at all
Proof that \((\lambda_i)\) given \((\mu_i)\) solve \(\sum_{i=1}^{N-1} \frac{\xi_i'}{z - \mu_i} = z + \mathcal{N}(0, \frac{2}{\beta}) : \quad \text{III}
\]

Ev. of \(M = E.v.\ of \ N'\). Those solve

\[
\det \begin{pmatrix}
\mu_1 - z & 0 & \cdots & 0 \\
0 & \mu_2 - z & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\frac{p_1}{\beta} & \frac{p_2}{\beta} & \cdots & \frac{p_{N-1}}{\beta}
\end{pmatrix} = 0
\]

Expanding in last column, we get

\[
\sum_{i=1}^{N-1} \frac{p_i \bar{p}_i}{\beta} \prod_{j \neq i} (\mu_j - z) + (q - z) \prod_{j=1}^{N} (\mu_j - z) = 0
\]

Dividing by \(\prod_{j=1}^{N} (\mu_j - z)\) and noticing that

\[
p_i \bar{p}_i \sim \frac{1}{e^\beta} \zeta^2 \beta, \quad \text{we are done}
\]
Interlacement of eigenvalues

Corollary 1. The eigenvalues of a matrix and its corner interlace:

\[ \lambda_1 \leq \mu_1 \leq \lambda_2 \leq \cdots \leq \mu_{N-1} \leq \lambda_N. \]

Proof. Using 2nd part of theorem (PSET 10), 
\[ \mu_i \text{ are roots of } f(z) = \sum_{i=1}^{N} \frac{z_i}{z - \lambda_i} = 0 \]

This is a polynomial equation of degree \( N-1 \) 
\( z_i > 0 \) almost surely \( \Rightarrow \) on the segment \([\lambda_i, \lambda_{i+1}]\) 
f(\( z \)) goes from \( +\infty \) to \( -\infty \). But \( f \) is continuous \( \Rightarrow \) 
\( \exists \) a root in each of the \((\lambda_i, \lambda_{i+1})\) segment and we located 
all roots \( \blacksquare \)
Corollary 2: The multilevel densities of $G\beta E$

Infinite matrix $X$ with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

All corners of $M = \frac{X+X^*}{2}$

\[
\begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

Joint density of interlacing eigenvalues.

\[
\prod_{k=1}^{N-1} \prod_{1\leq i<j\leq k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1} \exp \left( -\frac{\beta}{4} (x_i^N)^2 \right)
\]

Gaussian $\beta$ corners process

Gaussian potential
Corollary 3: \( \beta \)-corners processes

A self-adjoint matrix \( M \) whose law is **invariant** under \( M \mapsto UMU^* \)

\((U \text{ — orthogonal/unitary/symplectic if } \beta = 1, 2, 4)\)

Eigenvalues of corners

\[
\begin{pmatrix}
M_{11} & M_{12} & M_{13} & M_{14} \\
M_{21} & M_{22} & M_{23} & M_{24} \\
M_{31} & M_{32} & M_{33} & M_{34} \\
M_{41} & M_{42} & M_{43} & M_{44}
\end{pmatrix}
\]

Conditionally on \((x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N)\), the joint law is

\[
\prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^k \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}
\]

- A basis of extension from \( \beta = 1, 2, 4 \) to general \( \beta > 0 \).
- Consistent with Gaussian \( \beta \) Ensembles.
The multilevel process is a Markov chain and its law is computed through initial condition + transitions.

Either starting from level 0 and growing the level (works nicely for Gaussian $\beta$-corners) or starting from level $N$ and decreasing the level.
Sketch of the proof for multilevel densities (Corollaries 2 and 3)

If \( a_1 < a_2 < a_3 \) is fixed, then we compute the law as

\[
P_{3 \to 2} (\{a_1, a_2, a_3\} \to \{b_1, b_2\})
\]

\[
P_{2 \to 1} (\{b_1, b_2\} \to c_1)
\]

The transitions are given by the theorem. E.g., \((b_1, b_2)\) solve

\[
\frac{3}{2 - a_1} + \frac{3}{2 - a_2} + \frac{3}{2 - a_3} = 0.
\]

\(\star\)

We can renormalize \( w_i = \frac{\frac{3}{2 - a_i}}{\sum_{j \neq i} \frac{3}{2 - a_j}} \), so that \( \sum w_i = 1 \).

\(\star\) is unchanged
Then what remains is to make the change of variables to compute the density:

\[(w_1, w_2, w_3 \mid w_1 + w_2 + w_3 = 1) \leftrightarrow (b_1, b_2)\]

\[\frac{w_1}{\xi - a_1} + \frac{w_2}{\xi - a_2} + \frac{w_3}{\xi - a_3} = \frac{(\xi - b_1)(\xi - b_2)}{(\xi - a_1)(\xi - a_2)(\xi - a_3)}\]

The density of \((w_1, w_2, w_3)\) is explicit:

\[\frac{B/2-1}{w_1} \cdot \frac{B/2-1}{w_2} \cdot \frac{B/2-1}{w_3}\]

Dirichlet distribution ("Beta" on high-dimensional simplex)

because \(\chi^2_B \sim \text{density } x \leq 1, x > 0\).

Hence, need to change variables in density (PSET 1)
Conclusion: eigenvalues of corners of $\beta$ random matrices

Fix $\beta > 0$

$N = 1, 2, \ldots$

$a_1, \ldots, a_N \in \mathbb{R}$

**Definition.** Eigenvalues of corners of $N \times N$ random $\beta$-matrix with uniformly random eigenvectors and fixed eigenvalues $(a_i)^N_{i=1}$ are a triangular array $(x^k_i)_{1 \leq i \leq N}$ satisfying

$$x^k_{i+1} \leq x^k_i \leq x^k_{i+1}, \quad (x^N_1, \ldots, x^N_N) = (a_1, \ldots, a_N),$$

with distribution of density

$$\left[ \prod_{k=1}^N \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k} \right] \cdot \prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq k} (x^k_i - x^k_j)^{2-\beta} \prod_{a=1}^k \prod_{b=1}^{k+1} |x^k_a - x^{k+1}_b|^{|\beta/2 - 1|}.$$

What about $\beta = 0$ or $\beta = \infty$?
Theorem. With \((x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N)\), the eigenvalues with law

\[
\prod_{k=1}^{N-1} \prod_{1 \leq i < j \leq k} (x_i^k - x_j^k)^2 - \beta \prod_{a=1}^{k+1} \prod_{b=1}^{k} |x_a^k - x_b^k|^{\beta/2 - 1}
\]

converges as \(\beta \to \infty\) to the roots of derivatives:

\[
\prod_{i=1}^{k} (z - x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N} (z - a_j), \quad k = 1, 2, \ldots, N.
\]

Proof. \(\sum_{i=1}^{N} \frac{z_i}{z - x_i^N} = 0\). Given \(x_1^N, \ldots, x_N^N\), e.v. \(x_1^{N-1}, \ldots, x_{N-1}^{N-1}\) solve

\[
2z \sim \frac{1}{\beta} x^2 \quad \frac{1}{\beta} \sum_{i=1}^{N} N(0,1)^2 \to 1.
\]

By LLN for integer \(\beta\), by F/Nar computation for \(\beta \in \mathbb{R} \to \mathbb{C}\).

Hence, for \(\beta = \infty\) we get

\[
\sum_{i=1}^{N} \frac{1}{z - x_i^N} = 0.
\]

Multiplying by \(\prod_{i=1}^{N} (z - x_i^N)\), we see that these are roots of derivative.
One asymptotic result

**Theorem.** Suppose that as $N \to \infty$

$$\frac{1}{N} \sum_{i=1}^{N} \delta_{a_i/N} \to \mu,$$

with $G_{\mu}(z) = \int \frac{\mu(dx)}{z-x}$

$$\prod_{i=1}^{k} (z-x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N} (z-a_j)$$

and $k/N \to \alpha$.

Then

$$\frac{1}{k} \sum_{i=1}^{k} \delta_{x_i^k/k} \to \mu_{\alpha},$$

with $G_{\mu_{\alpha}}(z) = \int \frac{\mu_{\alpha}(dx)}{z-x}$

$$R_{\mu}(z) = (G_{\mu}(z))^{-1} - \frac{1}{z}, \quad R_{\mu_{\alpha}}(z) = (G_{\mu_{\alpha}}(z))^{-1} - \frac{1}{z}$$

"Voiculescu R-transform" "free projection" "free compression"

$\alpha R_{\mu_{\alpha}}(z) = R_{\mu}(z)$.

Same result for each $\beta > 0$, but **not** for $\beta = 0$. 
| End of Lecture 1. | Don’t forget about Problem set 1. |
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