General beta random matrix theory

(at MATRIX Institute)

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Lecture 1
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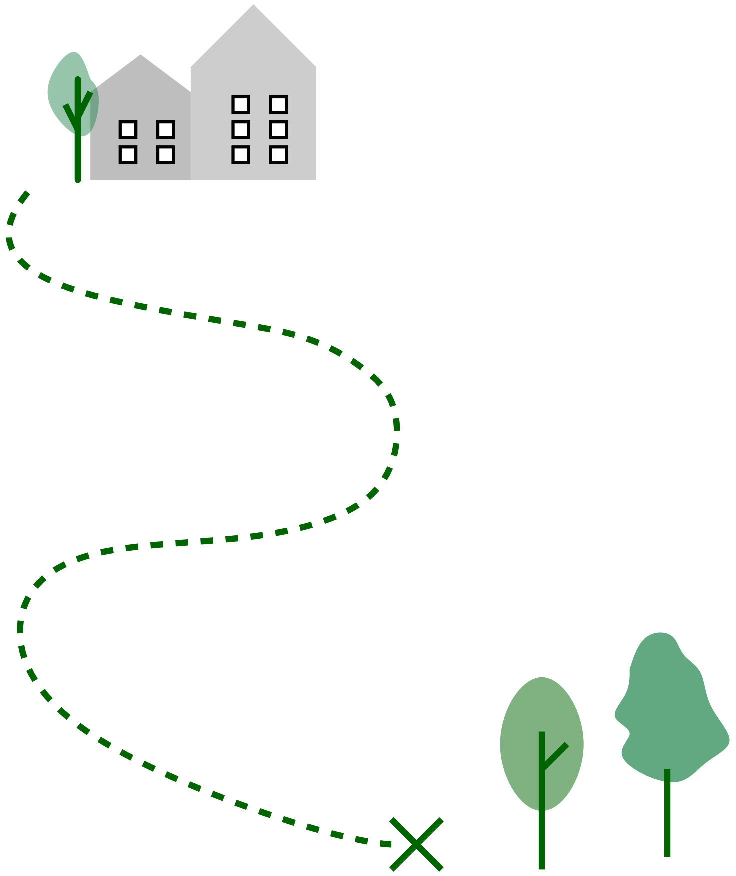
Roadmap



- What are general β random matrices?
- Lecture 1: corners of β random matrices.
- Problem set 1.
- Lecture 2: sums of β random matrices.
- Problem set 2.
- Lecture 3: questions and discussion of problem sets.

[EXCLUSIVE OFFER: Submit homework - receive a postcard!]

Lectures 1 and 2 are recorded, but Lecture 3 (office hours) is not!

This is NOT a research talk about brand new results.
Instead we explore **basic structures and definitions.**

(See "Lattice Paths, Combinatorics and Interactions" in 2 weeks).

Random matrix theory

The study of **random large** matrices and their eigenvalues.

Origins:

- Representation theory of the **classical groups** since 1920s.
  [Groups of matrices come with normalized measures.]
- Multidimensional **statistics** since 1930s.
  [Data is random and is naturally organized in 2-dimensional arrays.]
- Theoretical **physics** since 1950s.
  [Energy levels in heavy nuclei modelled by eigenvalues.]
- **Number theory** since 1970s.
  [Zeros of Riemann zeta-function modelled by eigenvalues.]
- Reemphasized in modern applied and statistical problems.
  ["Big data" revolution.]

The **central** and the **most basic** random matrix object is the **Gaussian Orthogonal/Unitary/Symplectic Ensemble.**

Gaussian β ensembles

$N \times N$ matrix X with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(\underset{\uparrow \text{ mean}}{0}, \underset{\uparrow \text{ variance}}{\frac{2}{\beta}})$.

$$M = \frac{X + X^*}{2} = \begin{pmatrix} M_{11} & M_{12} & \cdots \\ M_{21} & M_{22} & \\ \vdots & & \ddots \end{pmatrix}$$

(real!)

The density of **eigenvalues** $x_1 < x_2 < \cdots < x_N$:

$$\sim \prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\frac{\beta}{4}(x_i)^2\right).$$

$\beta = 1, 2, 4$ is the **dimension** of the base (skew-) field.

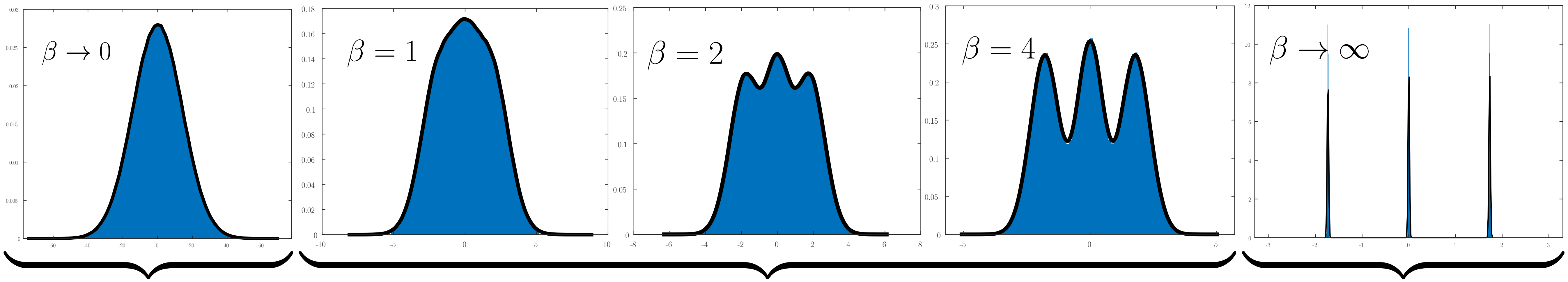
After today's lecture and pset you should be able to **prove it!**

Gaussian β ensembles

$$\prod_{1\leq i<j\leq N} (x_j - x_i)^\beta \prod_{i=1}^{N} \exp\left(-\tfrac{\beta}{4}(x_i)^2\right).$$

average of densities of 3 eigenvalues

First correlation function for $N = 3$: $\frac{1}{3}\mathbb{E}\left[\delta_{x_1} + \delta_{x_2} + \delta_{x_3}\right]$



i.i.d. Gaussian

random matrices

Hermite roots

Five meaningul values ask for a **unified treatment** of $\beta \in [0, +\infty]$

This is the topic of **general β random matrix theory.**

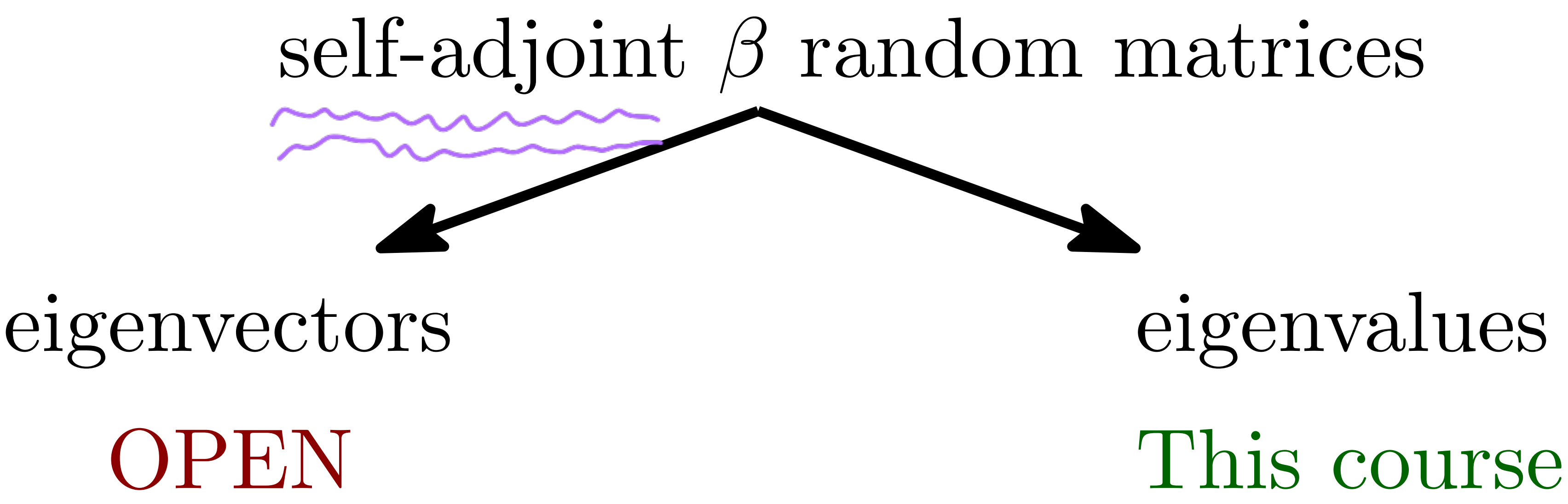
Tasks of β random matrix theory

- **Asymptotic questions:** E.g., $N \to \infty$ behavior of density

$$\prod_{1 \le i < j \le N} (x_j - x_i)^\beta \prod_{i=1}^{N} V(x_i).$$

with fixed $\beta > 0$, **or** $\beta \to 0$, **or** $\beta \to \infty$.

- **Algebraic questions:**
  How do we add and multiply general β random matrices?



self-adjoint β random matrices

eigenvectors

OPEN

eigenvalues

This course

Disclaimer: There is no field of dimension β.

Algebra: Rank 1 operations as a building block.

$$C = \underset{\text{arbitrary}}{\underset{\nearrow}{A}} + \underset{\text{rank 1 : a single non-zero eigenvalue}}{\underset{\nwarrow}{B}}$$

(Additive version)

Weyl's inequalities relate eigenvalues of A and C. The distributions are also the simplest

$$C = P \underset{\text{arbitrary } N \times N}{\underset{\nearrow}{A}} P \xleftarrow{} \text{rank } (N-1) \text{ projector}$$

(Multiplicative version)

If P projects on the first (N-1) coordinate vectors

$$A = \left(\begin{array}{c|c} C & \\ \hline & \end{array} \right)$$ — we cut corners

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Additive and multiplicative math are similar. We build our theore on the latter

The key computation

$N \times N$ matrix X with i.i.d. real/complex/quaternion Gaussian random variables with real parts $\mathcal{N}(0, \frac{2}{\beta})$. $M = \frac{X+X^*}{2}$.

$$\left(\begin{array}{ccc|c} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ \hline M_{41} & M_{42} & M_{43} & M_{44} \end{array}\right)$$

Eigenvalues:

- $(\lambda_i)_{i=1}^{N}$ — $N \times N$
- $(\mu_i)_{i=1}^{N-1}$ — $(N-1) \times (N-1)$

Theorem. Conditional distributions are:

1. (μ_i) given (λ_i) solve $$\sum_{i=1}^{N} \frac{\xi_i}{z - \lambda_i} = 0.$$
2. (λ_i) given (μ_i) solve $$\sum_{i=1}^{N-1} \frac{\xi_i'}{z - \mu_i} = z + \mathcal{N}(0, \tfrac{2}{\beta}),$$

ξ_i and ξ_i' are i.i.d. $\underline{\frac{1}{\beta}\chi^2_\beta}$ random variables, that is $\sum_{j=1}^{\beta} \mathcal{N}_j^2(0, \frac{1}{\beta})$.

particular case of Γ-random vars

Important: This is a basis of extension to all $\beta \in [0, +\infty]$.

Proof that (λ_i) given (μ_i) solve $\displaystyle\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \tfrac{2}{\beta})$: I

$$M = \begin{pmatrix} \boxed{A} & \begin{matrix} M_{1N} \\ \vdots \end{matrix} \\ M_{N1} & M_{NN} \end{pmatrix}$$

(We deal with $\beta = 2$ case)
$\beta = 1, 4$ is the same

A is self-adjoint $\Rightarrow \exists$ unitary matrix $U \in O(N-1)$

such that $U A U^* = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_N \end{pmatrix}$

Set $\tilde{U} = \begin{pmatrix} U & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$ and replace $M \to \underset{\displaystyle M'}{\underset{\|}{\hat{U} M \hat{U}^*}}$

Proof that (λ_i) given (μ_i) solve $\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$: II

we get

$$M' = \begin{pmatrix} \mu_1 & & 0 & p_1 \\ & \ddots & & \vdots \\ 0 & & \mu_{N-1} & p_{N-1} \\ \overline{p_1} & & \overline{p_{N-1}} & q \end{pmatrix}$$

Claim: Given $\mu_1, .., \mu_{N-1}$; $p_1, .. p_{N-1}$ are i.i.d. complex Gaussians, and q is a real $N(0, \frac{2}{\beta})$

Indeed, $(p_1, .. p_{N-1})$ obtained from last column of M by rotation with independent unitary matrix, which preserves being i.i.d. Gaussians.

q was not changed in $M \rightarrow UMU^*$ at all

Proof that (λ_i) given (μ_i) solve $\sum_{i=1}^{N-1} \frac{\xi_i'}{z-\mu_i} = z + \mathcal{N}(0, \frac{2}{\beta})$: III

E.V. of M = E.V. of M'. Those solve

$$\det \begin{pmatrix} \mu_1 - z & 0 & p_1 \\ & \ddots & \vdots \\ 0 & \mu_N - z & p_{N-1} \\ \bar{p}_1 \cdots & p_{N-1} & q - z \end{pmatrix} = 0$$

Expanding in last column, we get

$$-\sum_{i=1}^{N-1} p_i \bar{p_i} \cdot \prod_{j \neq i} (\mu_j - z) + (q - z) \prod_{j=1}^{N} (\mu_j - z) = 0$$

Dividing by $\prod_{j=1}^{N} (\mu_j - z)$ and noticing that

$p_i \overline{p_i} \sim \frac{1}{\beta} \chi^2_\beta$, we are done ▨

Interlacement of eigenvalues

Corollary 1. The eigenvalues of a matrix and its corner interlace:

$$\lambda_1 \le \mu_1 \le \lambda_2 \le \cdots \le \mu_{N-1} \le \lambda_N.$$

Proof. Using 2nd part of theorem (PSET 1?)

μ_i are roots of $f(z) = \sum_{i=1}^{N} \frac{\xi_i}{z-\lambda_i} = 0$

This is a polynomial equation of degree $N-1$

$\xi_i > 0$ almost surely $\Rightarrow$ on the segment $[\lambda_i, \lambda_{i+1}]$

$f(z)$ goes from $+\infty$ to $-\infty$. But f is continuous $\Rightarrow$

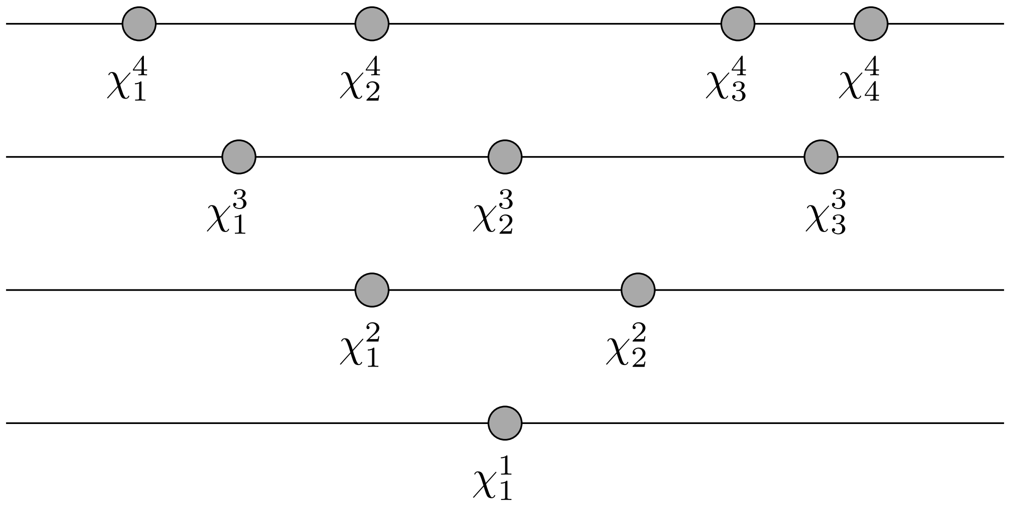
$\exists$ a root in each of the $(\lambda_i, \lambda_{i+1})$ segment and we located all roots ∎

Corollary 2: The multilevel densities of GβE

Infinite matrix X with i.i.d. real/complex/quaternion Gaussian random variables normalized so that their real parts are $\mathcal{N}(0, \frac{2}{\beta})$.

All corners of $M = \frac{X+X^*}{2}$

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



Joint density of **interlacing eigenvalues.**

cross-level interaction

$$\prod_{k=1}^{N-1}\prod_{1\le i<j\le k}(x_i^k-x_j^k)^{2-\beta}\prod_{a=1}^{k}\prod_{b=1}^{k+1}|x_a^k-x_b^{k+1}|^{\beta/2-1}\prod_{i=1}^{N}\exp\left(-\frac{\beta}{4}(x_i^N)^2\right)$$

in-level interaction

Gaussian β corners process

Gaussian potential

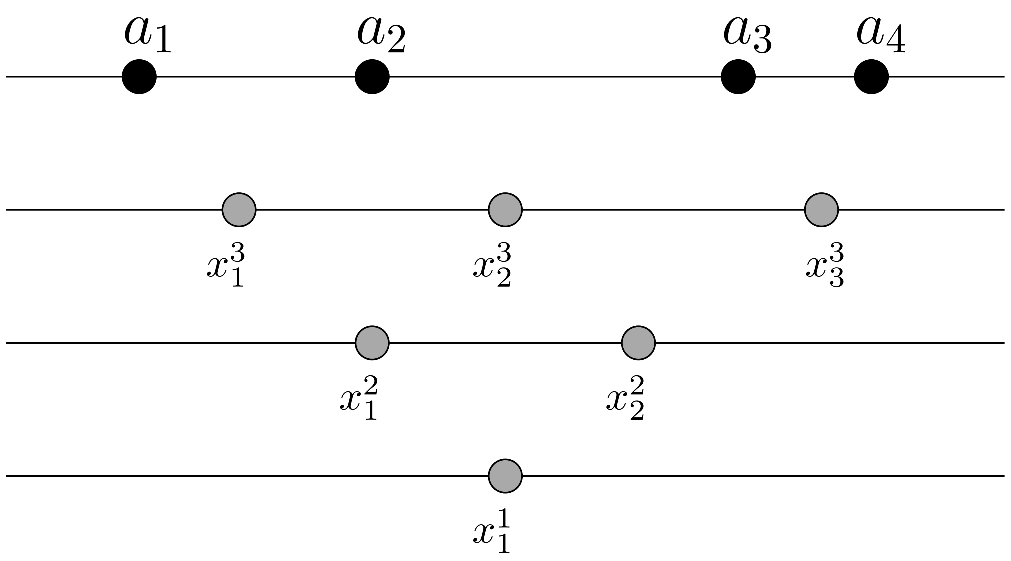
Corollary 3: β–corners processes

A self-adjoint matrix M whose law is **invariant** under $M \mapsto UMU^*$

(U — orthogonal/unitary/symplectic if $\beta = 1, 2, 4$)

Eigenvalues of corners

$$\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}$$



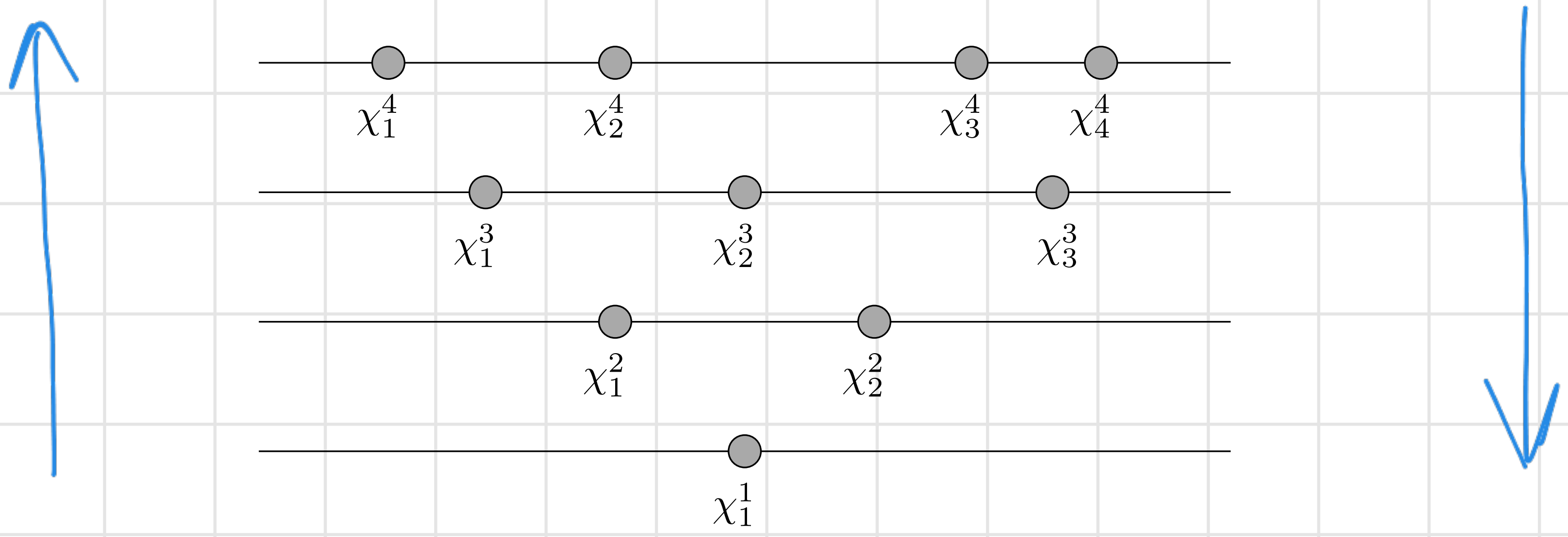
Conditionally on $(x_1^N, \ldots, x_N^N) = (a_1, \ldots, a_N)$, the joint law is

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

- A basis of extension from $\beta = 1, 2, 4$ to general $\beta > 0$.
- Consistent with Gaussian β Ensembles.

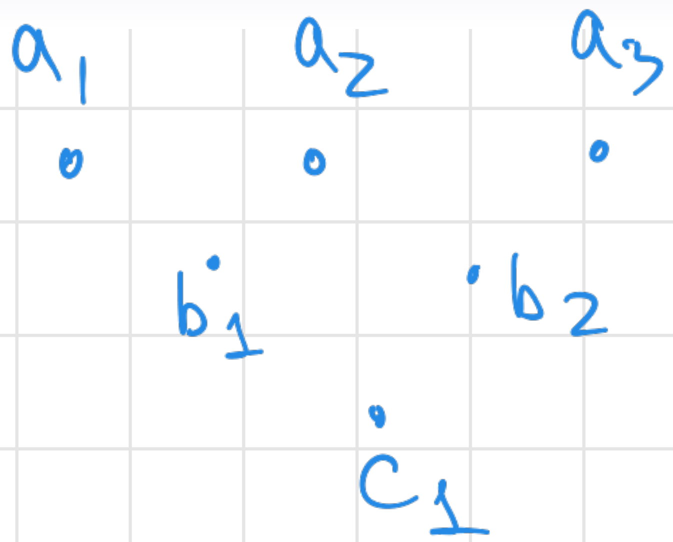
Sketch of the proof for multilevel densities (Corollaries 2 and 3) I

The multilevel process is a Markov chain and its law is computed through initial condition + transitions



Either starting from level 0 and growing the level (works nicely for Gaussian β-corners) or startring from level N and decreasing the level.

Sketch of the proof for multilevel densities (Corollaries 2 and 3) II



If $a_1 < a_2 < a_3$ is fixed,

then we compute the law

as $P_{3\to 2}\left((a_1,a_2,a_3) \to (b_1,b_2)\right) \cdot$

$\cdot P_{2\to 1}\left((b_1,b_2) \to c_1\right)$

The transitions are given by the theorem. E.g.

(b_1, b_2) solve $\frac{\xi_1}{z-a_1} + \frac{\xi_2}{z-a_2} + \frac{\xi_3}{z-a_3} = 0.$ (*'

We can renormalize $w_i = \frac{\xi_i}{\sum_j \xi_j}$, so that $\sum w_i = 1$

(*) is unchanged

Sketch of the proof for multilevel densities (Corollaries 2 and 3) III

Then what remains is to make the change of variables to compute the density:

$$(w_1, .. w_2, w_3 \mid w_1+w_2+w_3=1) \longleftrightarrow (b_1, b_2)$$

$$\frac{w_1}{z-a_1} + \frac{w_2}{z-a_2} + \frac{w_3}{z-a_3} = \frac{(z-b_1)(z-b_2)}{(z-a_1)(z-a_2)(z-a_3)}$$

The density of (w_1, w_2, w_3) is explicit:

$$w_1^{\beta/2-1} \cdot w_2^{\beta/2-1} \cdot w_3^{\beta/2-1}$$

„Dirichlet distribution"

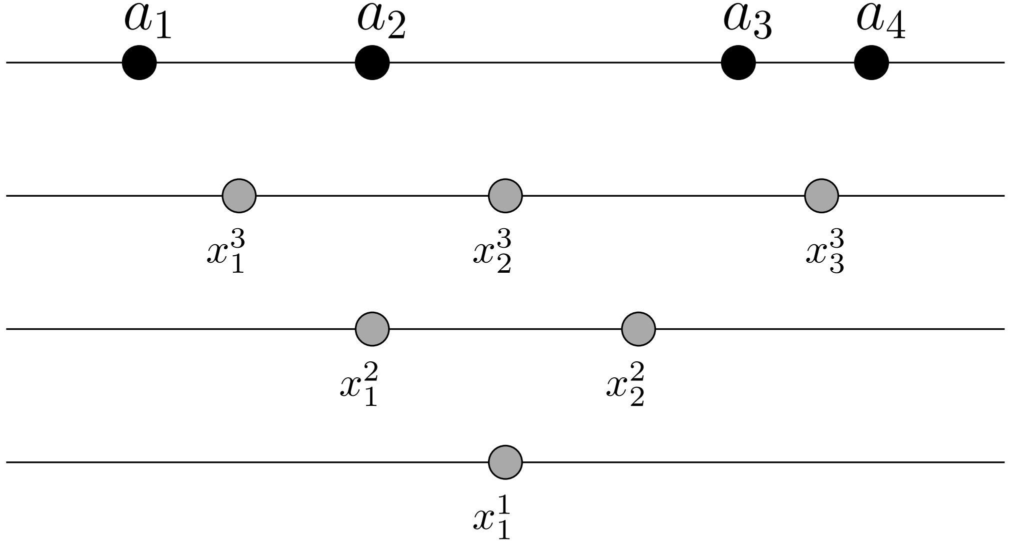
(„Beta" on high-dimensional simplex)

because $\chi^2_\beta \sim$ density $x^{\beta/2-1} e^{-x/2}, x>0$.

Hence, need to change variables in density (PSET 1)

Conclusion: eigenvalues of corners of β random matrices

Fix $\beta > 0$
$N = 1, 2, \dots$
$a_1, \dots, a_N \in \mathbb{R}$



Definition. Eigenvalues of corners of $N \times N$ **random β-matrix** with uniformly random eigenvectors and fixed eigenvalues $(a_i)_{i=1}^N$ are a triangular array $(x_i^k)_{1 \le i \le N}$ satisfying

$$x_{i+1}^k \le x_i^k \le x_{i+1}^{k+1}, \qquad (x_1^N, \dots, x_N^N) = (a_1, \dots, a_N),$$

with distribution of density

$$\left[\prod_{k=1}^{N} \frac{\Gamma(\frac{\beta k}{2})}{\Gamma(\frac{\beta}{2})^k}\right] \cdot \prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}.$$

What about $\beta = 0$ or $\beta = \infty$?

Theorem. With $(x_1^N, \dots, x_N^N) = (a_1, \dots, a_N)$, the eigenvalues with law

$$\prod_{k=1}^{N-1} \prod_{1 \le i < j \le k} (x_i^k - x_j^k)^{2-\beta} \prod_{a=1}^{k} \prod_{b=1}^{k+1} |x_a^k - x_b^{k+1}|^{\beta/2-1}$$

converges as $\beta \to \infty$ to the **roots of derivarives**:

$$\prod_{i=1}^{k} (z - x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N} (z - a_j), \qquad k = 1, 2, \dots, N.$$

Proof. Given $x_1^N, \dots x_N^N$, e.v. $x_1^{N-1}, \dots x_{N-1}^{N-1}$ solve

$$\sum_{i=1}^{N} \frac{\xi_i}{z - x_i^N} = 0 \qquad \xi_i \sim \frac{1}{\beta} \chi^2_\beta = \frac{1}{\beta} \sum_{i=1}^{\beta} N(0,1)^2 \xrightarrow[\beta\to\infty]{} 1$$

[By LLN for integer β, by $\mathbb{E}$/Var computation for $\beta \in \mathbb{R} \to \infty$]

Hence, for $\beta = \infty$ we get $\sum_{i=1}^{N} \frac{1}{z - x_i^N} = 0$.

Mutliplying by $\prod_{i=1}^{N} (z - x_i^N)$, we see that these are roots of derivative ∎

One asymptotic result

Theorem. Suppose that as $N \to \infty$

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{a_i/N} \to \mu, \qquad \text{with} \qquad G_\mu(z) = \int \frac{\mu(dx)}{z-x}$$

$$\prod_{i=1}^{k}(z - x_i^k) \sim \frac{\partial^{N-k}}{\partial z^{N-k}} \prod_{j=1}^{N}(z - a_j) \qquad \text{and} \qquad k/N \to \alpha.$$

Then

$$\frac{1}{k}\sum_{i=1}^{k} \delta_{x_i^k/k} \to \mu_\alpha, \qquad \text{with} \qquad G_{\mu_\alpha}(z) = \int \frac{\mu_\alpha(dx)}{z-x}$$

$$R_\mu(z) = (G_\mu(z))^{(-1)} - \frac{1}{z}, \qquad R_{\mu_\alpha}(z) = (G_{\mu_\alpha}(z))^{(-1)} - \frac{1}{z}$$

„Voiculescu R-transform"

„free projection"

„free compression"

$$\boxed{\alpha R_{\mu_\alpha}(z) = R_\mu(z)}.$$

Same result for each $\beta > 0$, but **not** for $\beta = 0$.

End of Lecture 1. Don’t forget about Problem set 1.

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