Dimers Statistical physics and probability
CUNY Vadim Gorin, TA: Matthew Nicoletti Discussion 2

Outline

Today we will review some questions about the discrete Loop Equations.

1. Non-intersecting walk transition probabilities.

Let $X(t) = (x_1, \ldots, x_N)$ denote the particle positions of the random walk corresponding to a uniformly random lozenge tiling of an $N \times N \times N$ hexagon at time $t \in [0, 2N] \cap \mathbb{Z}$. Show that for $\vec{e} \in \{0,1\}^N$, we have

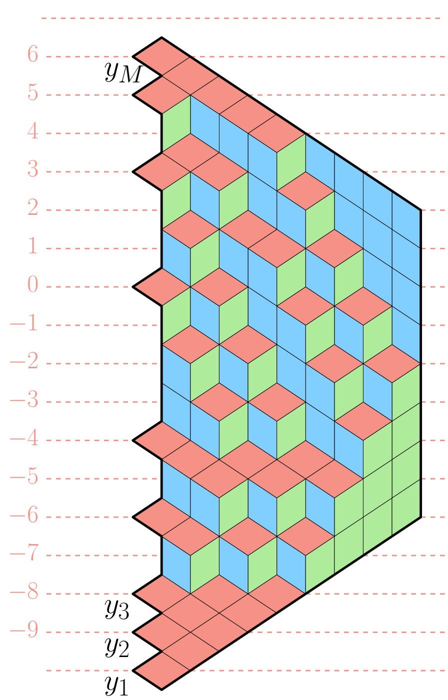
$$P\left(X(t+1) = \vec{x} + \vec{e} | X(t) = \vec{x}\right) \propto \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \cdot \prod_{i=1}^{N} \varphi_t^+(x_i)^{e_i} \varphi_t^-(x_i)^{1-e_i} \tag{1}$$

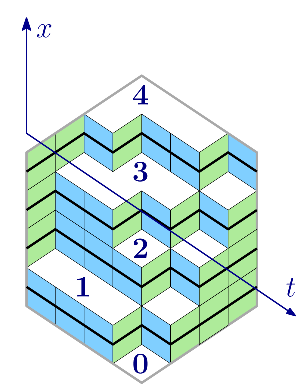
with $\varphi_t^+(x) = N - x - 1$ and $\varphi_t^-(x) = x + 2N - t$.

To prove this, use the fact that the number of Gelfand-Tsetlin patterns with fixed top row, is given by the Weyl dimension formula (c.f. Propp's lecture 1). A combinatorially equivalent fact is the following: The number of tilings of the trapezoidal region shown below, with the leftmost slice having M horizontal tiles at fixed positions $y_1 < y_2 < \cdots < y_M$, is given by

$$\prod_{1 \le i < j \le M} \frac{y_i - y_j}{i - j}.$$

Note that the positions of random walk paths are complementary to the positions of horizontal tiles.





2. Dynamical loop equations.

Recall from Lecture 2 the general measure (which may be complex valued, but is normalized so that the sum of "probabilities" is 1) defined by

$$P(\vec{e}) \propto \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \cdot \prod_{i=1}^{N} \varphi^+(x_i)^{e_i} \varphi^-(x_i)^{1-e_i}$$

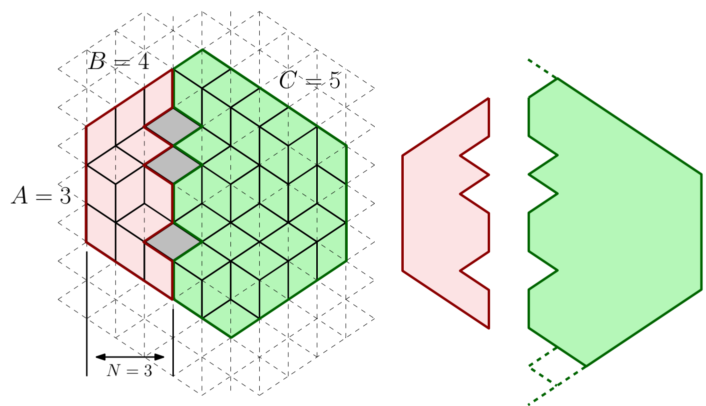
where $\frac{1}{N}x_1, \dots, \frac{1}{N}x_N$ are contained in a domain $\mathscr{D} \subset \mathbb{C}$, and $\varphi^\pm(Nz)$ are holomorphic functions of z on $\mathscr{D}$. Show that

$$R_N(z) = \mathbb{E}\left[\varphi^+(z) \prod_{i=1}^{N} \frac{z - x_i + (1 - e_i)}{z - x_i} + \varphi^-(z) \prod_{i=1}^{N} \frac{z - x_i - e_i}{z - x_i}\right]$$

is holomorphic on $N\mathscr{D}$, by directly checking the the residue at each possible pole is 0. (Recall that in lecture we showed this by writing it as a ratio of two partition functions, which were both holomorphic functions).

3. Non-intersecting walk distribution at a fixed time (lozenges along a vertical slice). Let us return to the setting of uniformly random lozenge tilings of an $N \times N \times N$ hexagon. Derive a formula for the marginal distribution of the positions $\vec{x} = (x_1, \dots, x_N)$ of the corresponding non-intersecting walks at time t.

Hint: Non-intersecting path positions at time t are determined by positions of horizontal lozenges on the time t slice. To compute the probability of a fixed configuration of horizontal lozenges on a slice, use the Weyl dimension formula to count ways to fill in the tiling in the part of the hexagon to the left and to the right of the slice. In the figure below we illustrate this procedure in the slightly more general setting of a hexagon with unequal side lengths A, B, C.



Hint 2: The answer, in terms of horizontal lozenges $y_1 < \cdots < y_t$ at the time t slice (in the coordinate system illustrated in the right figure of problem 1), should be

$$\frac{1}{Z} \prod_{1 \le i < j \le t} (y_i - y_j)^2 \prod_{i=1}^{t} w_t(y_i + N) \tag{2}$$

where Z is a normalization. Above $w_t(y) = (y+1)_{N-t}(N+t-y)_{N-t}$. here we use the notation for Pochhammer symbols $(a)_i = a(a+1)\cdots(a+i-1)$.

The formula (2) can be written in terms of the random walk positions $(x_1, \dots, x_N)$ at time t as

$$\frac{1}{Z} \prod_{1 \le i < j \le N} (\tilde{x}_i - \tilde{x}_j)^2 \prod_{j=1}^{N} \frac{(\alpha+1)_{\tilde{x}_i} (\beta+1)_{M-\tilde{x}_i}}{\tilde{x}_i!(M - \tilde{x}_i)!}$$

where α, β, M are parameters depending on N and t, and $\tilde{x}_i$ differ from x_i by a shift, so that $\tilde{x}_i \in \{0, 1, \dots, M\}$.

4. Nekrasov's equation. Note that with w_t as in (2) above,

$$\frac{w_t(x)}{w_t(x-1)} = \frac{x+N-t}{x}\frac{N+t-x}{2N-x} = \frac{\phi^+(x)}{\phi^-(x)}$$

if we define

$$\phi^+(z) = (z+N-t)(N+t-z) \qquad\qquad \phi^-(z) = z(2N-z).$$

Let $\{y_1 < y_2 < \cdots < y_t\} \subset \{0,1,\dots,N+t-1\}$ be sampled randomly from the distribution

$$\mathbb{P}(y_1,\dots,y_n) = \frac{1}{Z}\prod_{1\le i<j\le t}(y_i-y_j)^2\prod_{i=1}^{t} w_t(y_i)$$

(note that here we have absorbed the shift by N into the y_i, as compared to (2)).

Let

$$R_{N,t}(z) = \mathbb{E}\left[\phi^-(z)\prod_{i=1}^{t}\left(1-\frac{1}{z-y_i}\right)+\phi^+(z)\prod_{i=1}^{t}\left(1+\frac{1}{z-y_i-1}\right)\right].$$

Using the same strategy as in question 2, show that $R_{N,t}(z)$ is holomorphic.

5. Limit shape for lozenge tilings of a hexagon.

Using the result of Question 4, compute the limit shape, assuming that the corresponding law of large numbers holds, for uniformly random tilings of large $N\times N\times N$ hexagons. More precisely, assume that the empirical Stieltjes transforms $G_{N,\tau}(z) = \int \frac{1}{z-x}\mu_{N,N\tau}(dx)$, where $\mu_{N,t} = \frac{1}{t}\sum_{j=1}^{t}\delta_{y_j/N}$ is the density of the rescaled (by $1/N$) positions of horizontal lozenges at time $t = \lfloor N\tau\rfloor$, have a limit, which we call $G_\tau(z)$. Then, one may follow the following steps:

- First show that

$$\mathbb{E}\left[\prod_{i=1}^{t}\left(1-\frac{1}{N(z-y_i/N)}\right)\right] = \exp(-\tau G_\tau(z))(1+o(1)). \tag{3}$$

  Show a similar asymptotic for the second term in the definition of $R_{N,t}$.

- Compute the limit

$$R_\tau(z) = \lim_{N\to\infty}\frac{1}{N^2}R_{N,N\tau}(Nz)$$

  in two ways. Compute it in one way by noting that $R_\tau(z)$ is a holomorphic function, and that in fact it is a degree two polynomial. And in another way using (3).

- Equate the two expressions for R_τ. This gives a degree two polynomial equation solved by $\exp(\tau G_\tau(z))$. Use this to solve for $G_\tau(z)$. Note that the limiting horizontal tile density $\mu_\tau(x)$ is related to the limiting height function h (in the discrete setting, the height function is shown in the bottom right of question 1) by $\partial_x h(\tau,x-1) = 1-\mu_\tau(x)$. Here the $x-1$ is because we are transforming back to the coordinates shown in the bottom right of question 1.