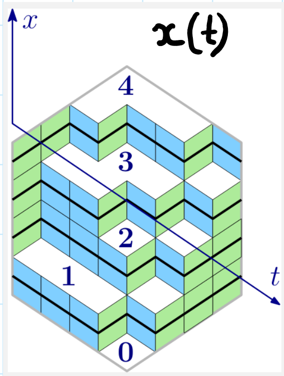
Lecture 3: Burgers equation and GFF

Friday, August 18



Notations

Theorem 0 (last lecture):

$$\rho_t(s) = \sum_{i=1}^{N} \mathbb{1}\left(\frac{x_i(t)}{N} \le s \le \frac{x_i(t)}{N} + \frac{1}{N} \right) \quad \text{density of}$$

$$G_t(z) = \exp\left(\int \frac{\rho_t(s)}{z-s}\, ds \right) \qquad F_t(z) = -\frac{\varphi_t^+(z)}{\varphi_t^-(z)} G_t(z) = \exp\left(- \int_{\frac{t}{N}-2}^{1-\frac{1}{N}} \frac{1-\rho_t(s)}{z-s}\, ds \right)$$

$$B_t(z) = \varphi_t^-(z)\left[F_t(z) - 1 \right] \qquad \varphi_t^+(z) = 1 - z - \frac{1}{N} \qquad \varphi_t^-(z) = z + 2 - \frac{t}{N}$$

Conclusion:

$$N \int \frac{\rho_{t+1}(s) - \rho_t(s)}{z-s}\, ds = \frac{1}{2\pi i} \oint \frac{\ln B_t(w)}{(w-z)^2}\, dw + \frac{1}{N}\left(\begin{matrix}\text{explicit}\\ \text{deterministic}\end{matrix} \right) + \Delta M_t(z) + O\left(\frac{1}{N^2} \right)$$

$$\text{(D)} \qquad E\left[N \Delta M_t(z_1) \Delta M_t(z_2) \,\middle|\, x(t) \right] = \frac{1}{2\pi i} \oint \frac{G_t(w)}{F_t(w) - 1} \frac{dw}{(w-z_1)^2 (w-z_2)^2}$$

Assumption we have not checked yet: $\oint_{\text{large}} \frac{\partial_z B_t(z)}{B_t(z)}\, dz = 0$ | $\exists$ branch of $\ln B_t(z)$

Follows from: I) No zeros of $B_t(z)$ outside contour; II) $B(z) = C_0 + \frac{C_1}{z} + \dots$ as $z \to \infty$, $C_0 \neq 0$

I) $\Downarrow$ $\oint$ = residue at ∞
[$F_t(z) \neq 1 \iff \int \frac{1-\rho}{z-s} ds \neq 0$, which is checked directly by looking at Im and Re]

II) $\Downarrow$ this residue is 0
[directly from definition]

Task: solve (D) as $N \to \infty$.

Theorem 1) As $N \to \infty$ $\rho_{Nt} \longrightarrow$ deterministic density „limit shape"

described in terms of $f_t(z) = \lim_{N \to \infty} F_{tN}(z)$, $0 \le t \le 2$, $z \in \mathbb{C} \setminus [t-2, 1]$

which satisfies: $\partial_t \ln f_t(z) + \partial_z \ln\left(1 - f_t(z)\right) = 0$
(for all $0 < t < 2$, $z \in \mathbb{C} \setminus [t-2, 1]$)

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Limit shapes and the complex Burgers equation ← first appearance of such eq. in tilings

Limit shapes and the complex Burgers equation

by

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← first appearance of such eq. in tilings

Difference: KO had $z \in \mathbb{R}$, we have $z \in \mathbb{C}$

Proof| Take (D), divide by N, sum from 0 to $tN-1$

Summing the variances, random disappears as $N \to \infty$

Differentiating in t, get a DE for the density:

$$\frac{\partial}{\partial t} \lim_{N\to\infty} \int \frac{\rho_{Nt}(s)}{z-s}\, ds = \frac{1}{2\pi i} \oint \frac{\ln(1-f_t(w)) + \ln(w+2-t)}{(z-w)^2}\, dw =$$

Deform the contour to be large. $\mathrm{Res}(w=\infty)=0$.

The residue at $z=w$ is derivative

$$= -\frac{\partial}{\partial z} \ln(1-f_t(z)) - \frac{1}{z+2-t}$$

cancels.

But from definition

$$\frac{\partial}{\partial t} \ln f_t(z) = \lim_{N\to\infty} \frac{\partial}{\partial t} \int \frac{\rho_{Nt}(s)}{z-s}\, ds + \frac{1}{z+2-t}$$

We get $\frac{\partial}{\partial t} \ln f_t(z) + \frac{\partial}{\partial z} \ln(1-f_t(z)) = 0$. $\boxed{*}$

How do you solve ↑ ? Method of characteristics

(*) $$\frac{\partial_t f}{f} + \frac{\partial_z f}{f-1} = 0$$

Consider a curve $(t(\tau), z(\tau), f(\tau))$ solving

(**) $\partial_\tau t = \frac{1}{f}$ $\quad \partial_\tau z = \frac{1}{f-1}$ $\quad \partial_\tau f = 0$

[curve in $1+2+2=5$ dimensional space]

If f solves, then $f_{t(\tau)}(z(\tau))$ solves (**)

And other way round: curves solving (**) can be glued together into a graph of solution to (*)

In principle, curves (**) might be complicated, but in our case we can use $\boxed{t=\tau}$ as a parameter

$$(***)\quad \begin{cases} \partial_t z(t) = \dfrac{f}{f-1}, \\ f(t) = \text{const} = \underbrace{f(0)}_{\text{we know it!}} \end{cases} \Rightarrow \quad z(t) = z(0) + t\cdot\frac{f}{f-1}$$

Algorithm for getting $f_t(z)$: $z\in\mathbb{C}^+$, $f\in\mathbb{C}^+$:

$$f_0(u) = \frac{u^2-1}{u(u+2)}$$ | computed by integrating $\rho_0 = \mathbb{1}_{-1\le s\le 0}$

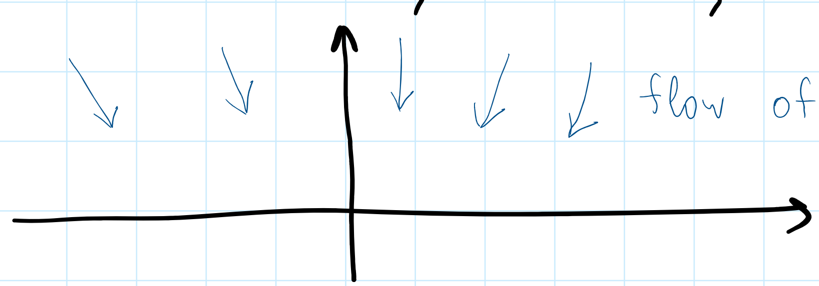
Find $u\in\mathbb{C}^+$ such that:

$$(\mathrm{DD})\qquad \underbrace{z}_{z(t)} = \underbrace{u}_{z(0)} + t\cdot\frac{f_0(u)}{f_0(u)-1}$$

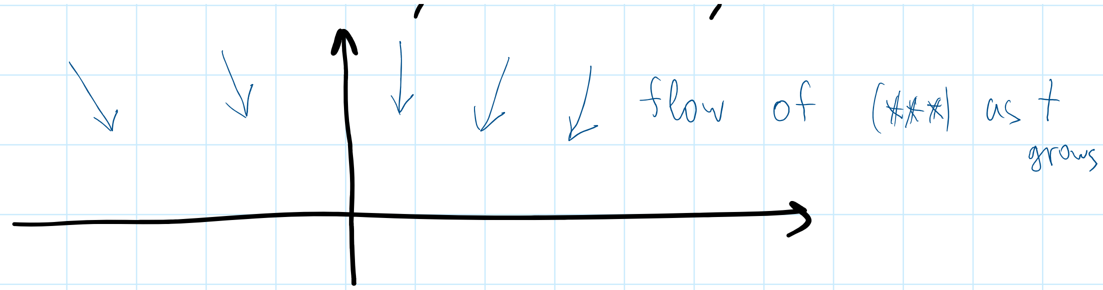
| This is $(2-t)u^2 + (1-2z)u + (t-z) = 0$ [Lecture 1 equation]

Solve the equation and set $f_t(z) = f_0(u)$. (DDD)

Lemma: 1) For each $0\le t\le 2$, $z\in\mathbb{C}^+$, $\exists\, u\in\mathbb{C}^+$ solving



flow of (***) as t grows



2) For real $z = x$, (DD) is the equation from Lecture 1 giving the map Ω (bijection of the liquid region with $\mathbb{C}^+$).

Giving a geometric description of Ω:
Start from a point in $\mathbb{C}^+$, run the flow $z(t)$ until it hits the real axis at time t, position x. (t, x) is the point of the liquid region corresponding to $\mathbb{C}^+$-point you started from.

3) (DDD) is the complex slope of Lecture 1.

Exercise: finish the proof of Lemma.

Fluctuations and GFF.

Proposition 1: $\mathcal{L}_t(z) = \int \frac{p_t(s)}{z-s}\, ds - \mathbb{E} \int \frac{p_t(s)}{z-s}\, ds$

As $N \to \infty$,

$$N\left(\mathcal{L}_{t+1}(z) - \mathcal{L}_t(z)\right) = \partial_z\left(\mathcal{L}_t(z)\, \frac{f_{t/N}(z)}{1 - f_{t/N}(z)}\right) \qquad (\Delta)$$

$$+ \underbrace{\Delta \mathcal{M}_t(z)}_{\text{from Theorem 0.}} + \text{(smaller orders)}$$

Discrete stochastic PDE for $\mathcal{L}_t(z)$

Proof| Take (D) of Theorem 0, subtract its expectation, Taylor expand the 1st term in RHS ∎

Proposition 2| (Δ) can be solved along the characteristics: Let $z(t,u) := z^t$ solving $\begin{cases} \partial_t z^t = \frac{f_0(u)}{f_0(u)-1}, \\ z^0 = u. \end{cases}$

Then (D) is rewritten as ($\Delta\Delta$)

$$N\left(\mathcal{L}_{tN+1}\left(z\left(t+\tfrac{1}{N},u\right)\right)\partial_u z\left(t+\tfrac{1}{N},u\right) - \mathcal{L}_{tN}\left(z(t,u)\right)\partial_u z(t,u)\right)$$

$$= \Delta\mathcal{M}_{tN}\left(z(t,u)\right)\partial_u z\left(t+\tfrac{1}{N},u\right) + (\text{smaller orders})$$

Proof: Direct substitution ∎

($\Delta\Delta$) is straightforward to solve by summing it.

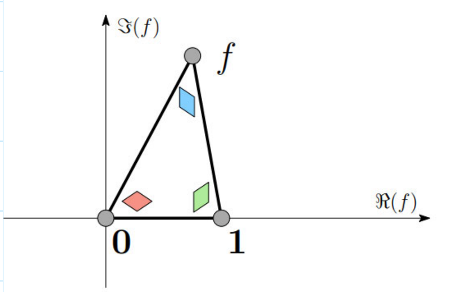
Exercise: Use Prop. 2 to finish the proof of GFF – CLT of Lecture 1
(or Gorin – Huang, Section 6)

Conclusions| In computations for LLN and GFF

the PDE $\partial_t \ln f_t(z) + \partial_z \ln(1 - f_t(z)) = 0$
played role, as well as its characteristics.

In our approach, taking $z \in \mathbb{C}$ was important

If $z \in \mathbb{R}$ instead, then you get complex slope of lecture 1,



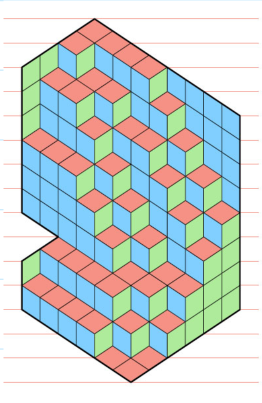
Triangle $(0,1,f)$ has angles $(\pi p_{\diamond}, \pi p_{\square}, \pi p_{\square})$.

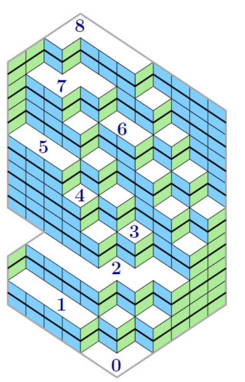
$$\exp\Big(-\int \frac{1-p(s)}{\underset{\shortparallel}{s-x+i0}}\,ds\Big)$$

complex number with argument

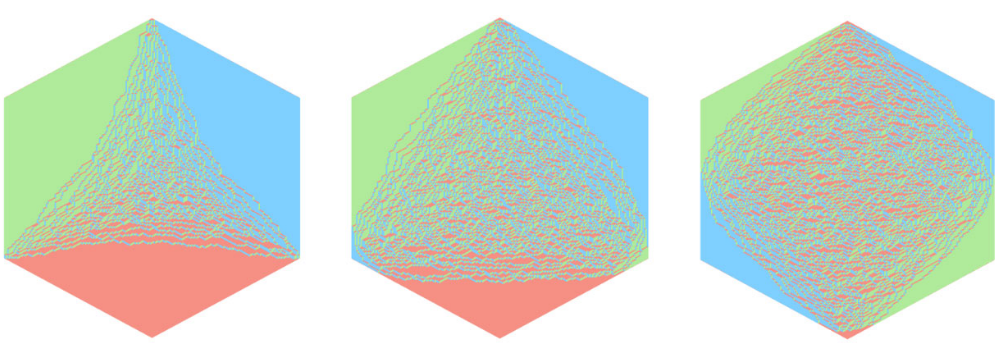
$$-\pi\,(\underbrace{1-p(x)}_{\text{density of } \diamond})$$

Technology works for: more advanced domains





More advanced probability measures



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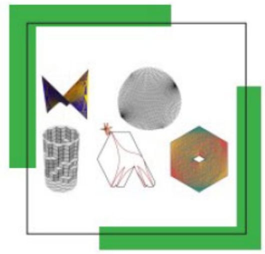

March 11 - June 14, 2024

Long Program Schedule

- Opening Day: March 11, 2024
- Geometry, Statistical Mechanics, and Integrability Tutorials: March 12-15, 2024
- Workshop I: Statistical Mechanics and Discrete Geometry: March 25-29, 2024
- Workshop II: Integrability and Algebraic Combinatorics: April 15-19, 2024
- Workshop III: Statistical Mechanics Beyond 2D: May 6-10, 2024
- Workshop IV: Vertex Models: Algebraic and Probabilistic Aspects of Universality: May 20-24, 2024
- Culminating Workshop at Lake Arrowhead: June 9-14, 2024

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