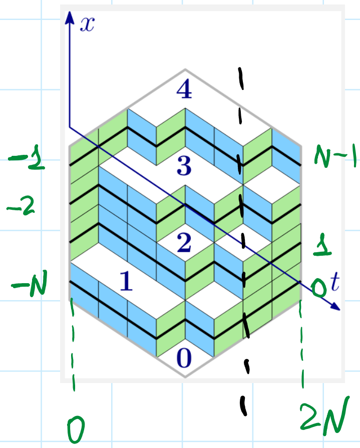
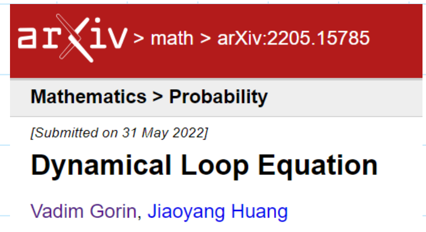
Lecture 2: Dynamical Loop Equation

Wednesday, August 16





$X(t)$, $t = 0, 1, \ldots 2N$

$X(0) = (-1, -2, \ldots -N)$

$X(2N) = (N-1, N-2, \ldots, 0)$

Uniformly random choice

Proposition 1: $X(t)$ is a Markov chain with transition probabilities

$$\mathrm{Prob}\big(X(t+1) = \vec{x} + \vec{e} \mid X(t) = \vec{x}\big) \sim \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \prod_{i=1}^{N} \left[\varphi_t^+(x_i)\right]^{e_i} \left[\varphi_t^-(x_i)\right]^{1-e_i}$$

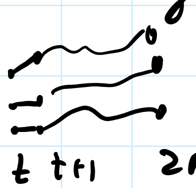
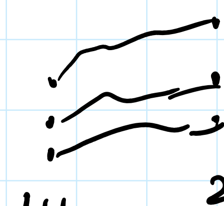
$\vec{e} \in \{0,1\}^N$

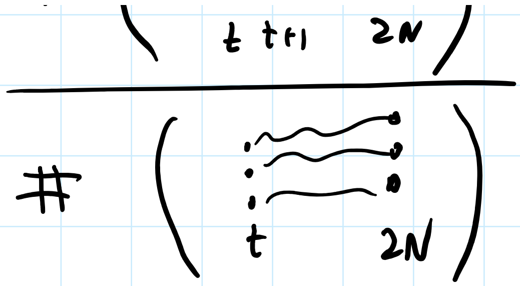
$$\varphi_t^+(x) = \frac{N-x-1}{N} \qquad \varphi_t^-(x) = \frac{x+2N-t}{N}$$

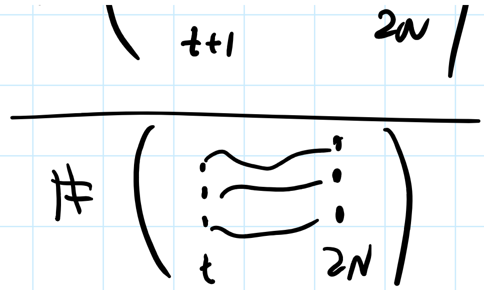
Sketch of the proof: 1) Markov property

" Given present, past and future are independent"
straightforward from uniformity.

2) Transition probability

Prob = # () / ... = # ()

Prob = 

= 

Both numerator and denominator are counted as # Gelfand-Tsetlin patterns in Propp's lecture 1.

[Problem set for details].

Key tool

Theorem 1 (dynamical loop equation)

Consider a prob. measure on $\vec{e} \in \{0,1\}^N$

$$\mathrm{Prob}(\vec{e}) \sim \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \cdot \prod_{i=1}^{N} \varphi^+(Nx_i)^{e_i} \, \varphi^-(Nx_i)^{1-e_i}$$

$x_i \in$ a domain $\mathcal{D} \in \mathbb{C}$ and $\varphi^\pm(z)$ are holomorphic in z

Consider $R(z) = \mathbb{E}\left[\varphi^+(z) \prod_{i=1}^{N} \frac{z - x_i + (1-e_i)}{z - x_i} + \varphi^-(z) \prod_{i=1}^{N} \frac{z - x_i - e_i}{z - x_i}\right]$

Then $R(z)$ is holomorphic in $\mathcal{D}$ (= has no singularities).

Proof $Z_N(x_1, \dots x_N) = \sum_{\vec{e} \in \{0,1\}^N} \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \prod_{i=1}^{N} \varphi^+(Nx_i)^{e_i} \, \varphi^-(Nx_i)^{1-e_i}$

We claim than Z_N is holomorphic in $x_1, \dots x_N \in \mathcal{D}$

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Write $Z_N = \dfrac{1}{\prod_{i<j}(x_i - x_j)} \underbrace{\sum_{\vec{e}} \prod_{i<j} \left(x_i + e_i - (x_j + e_j)\right) \prod_{i=1}^{N} \varphi^+(Nx_i)\, \varphi^-(Nx_i)^{1-e_i}}_{\text{skew-symmetric in } x_i}$

vanishes on $x_i = x_j$ line

Hence, you can divide by all $(x_i - x_j)$ factors

$$\frac{Z_{N+1}(x_1, \dots x_N, z)}{Z_N(x_1, \dots x_N)} = \sum_{e_{N+1} \in \{0,1\}} \prod_{i=1}^{N} \frac{(x_i + e_i) - (z + e_{N+1})}{x_i - z} \varphi^+(Nz)^{e_{N+1}} \varphi^-(Nz)^{1-e_{N+1}}$$

$$\cdot \frac{1}{Z_N} \sum_{e_1 \dots e_N} \prod_{i<j} \frac{(x_i + e_i) - (x_j + e_j)}{x_i - x_j} \prod_{i=1}^{N} \varphi^+(Nx_i)^{e_i} \varphi^-(Nx_i)^{1-e_i}$$

$$= R(z).$$

$\blacksquare$

How is this helpful?

$$\rho(s, \vec{x}) = \sum_{i=1}^{N} \mathbb{1}\left(\frac{x_i}{N} \le s \le \frac{x_i}{N} + \frac{1}{N}\right)$$

particle configuration

$\rho(s)\, ds$ — probability measure

Smoothed empirical measure $\quad \int_{-\infty}^{y} \rho(s, \vec{x})\, ds \to$ Height function at y.

$$G(z) = \exp\left(\int \frac{\rho(s, \vec{x})}{z - s}\, ds\right) \quad \Big| \, z \in \mathbb{C}$$

Stieltjes transform

$$B(z) = \varphi^+(z)\, G(z) + \varphi^-(z)$$

Theorem 2: $\operatorname{Prob}(\vec{e}) \approx \prod \dfrac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \prod \varphi^+(x_i)^{e_i} \varphi^-(x_i)^{1-e_i}$

Theorem 2: $\mathrm{Prob}(\vec{e}) \doteq \prod_{i<j} \frac{x_i + e_i - (x_j + e_j)}{x_i - x_j} \prod \varphi^+(x_i)^{e_i} \, \varphi^-(x_i)^{1-e_i}$

Assume: $\left\{ \frac{x_i}{N} \right\} \in [\ell, r]$ some segment

$$\oint_{\text{around } [\ell, r]} \frac{\partial_z B(z)}{B(z)} dz = 0$$

means that $\ln(B(z))$ has a single-valued branch outside $[\ell, r]$

Then $N \int \frac{\rho(s; \vec{x}+\vec{e}) - \rho(s, \vec{x})}{z - s} ds = \frac{1}{2\pi i} \oint \frac{\ln B(w)}{(w-z)^2} dw$

$+ \frac{1}{N} \left(\begin{matrix} \text{Not shown,} \\ \text{explicit,} \\ \text{deterministic} \end{matrix} \right) + \Delta \mathcal{M}(z) + O\left(\frac{1}{N^2}\right)$

$\Delta \mathcal{M}(z)$ — Gaussian process with covariance (mean 0)

$$\mathbb{E}\left[N \Delta \mathcal{M}(z_1) \Delta \mathcal{M}(z_2)\right] = \frac{1}{2\pi i} \oint \frac{G(w)\, \varphi^+(w)}{B(w)} \frac{dw}{(w-z_1)^2 (w-z_2)^2}$$

Evolution equation for $\rho(s, \vec{x})$

Full proof: Th 4.5 in Gorin – Huang

Today:
1) why assumption is important
2) why is it true?

$$A(z) = \mathbb{E} \prod_{i=1}^{N} \frac{z - \frac{1}{N}(x_i + e_i)}{z - \frac{x_i}{N}} = \underbrace{\mathbb{E} \prod_{i=1}^{N} \left(1 - \frac{1}{N} \frac{e_i}{z - \frac{x_i}{N}}\right)}_{\exp\left(-\frac{1}{N} \sum \frac{e_i}{z - \frac{x_i}{N}} + \dots\right)}$$

$$\exp\left(-\frac{1}{N}\sum_{i=1}\frac{\ell_i}{z-\frac{x_i}{N}}+\dots\right)$$

$\approx$ what we want to compute

We find asymptotics of $A(z)$.

$$R\left(\frac{z}{N}\right)=\varphi^+(z)\,\mathbb{E}\prod_{i=1}^{N}\left(1+\frac{1}{N}\frac{1-\ell_i}{z-\frac{x_i}{N}}\right)+\varphi^-(z)\,\underbrace{\mathbb{E}\prod\left(1-\frac{1}{N}\frac{\ell_i}{z-\frac{x_i}{N}}\right)}_{A(z)}$$

$$\approx \mathbb{E}\left[\prod\left(1+\frac{1}{N}\frac{1}{z-\frac{x_i}{N}}\right)\prod\left(1-\frac{1}{N}\frac{\ell_i}{z-\frac{x_i}{N}}\right)\right]$$

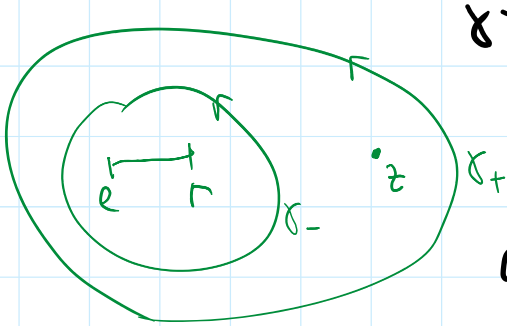
$$\approx A(z)\cdot G(z)+\underline{O}\left(\frac{1}{N}\right)$$

$$\underbrace{R\left(\frac{z}{N}\right)}_{\text{holomorphic}}=\underbrace{A(z)}_{\text{unknown}}\cdot\underbrace{B(z)}_{\text{explicit}}+\underline{O}\left(\frac{1}{N}\right)$$

Lemma Under assumption, we have

$$\ln A(z)=\frac{1}{2\pi i}\oint_{\gamma^-}\frac{\ln B(w)}{w-z}\,dw+\underline{O}\left(\frac{1}{N}\right)$$

Proof



The game is about different singularities for R and A

$$\ln B(z)=h_+(z)-h_-(z) \qquad h_-(z)=\frac{1}{2\pi i}\oint\frac{\ln B(w)}{w-z}\,dw$$

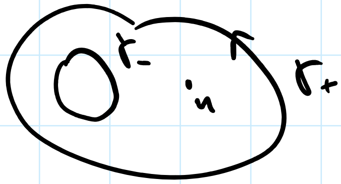
$$\ln B(z) = h_+(z) - h_-(z)$$

$$h_-(z) = \frac{1}{2\pi i} \oint_{\gamma-} \frac{\ln B(w)}{w-z}\, dw$$

$$h_+(z) = \frac{1}{2\pi i} \oint_{\gamma+} \frac{\ln B(w)}{w-z}\, dw$$

[residue inside the contour]

$$\underbrace{R\left(\tfrac{z}{N}\right) \exp\left(-h_+(z)\right)}_{\text{holomorphic in } z \text{ inside } \gamma_+} = A(z) \exp\left(-h_-(z)\right) + O\left(\tfrac{1}{N}\right)$$



$$0 = \frac{1}{2\pi i} \oint_{\gamma-} \frac{R\left(\frac{z}{N}\right) \exp\left(-h_+(z)\right)}{z-u}\, dz =$$

$$= \frac{1}{2\pi i} \oint_{\gamma-} \frac{A(z) \exp\left(-h_-(z)\right)}{z-u}\, dz + O\left(\tfrac{1}{N}\right)$$

$$= A(u) \exp\left(-h_-(u)\right) + \underbrace{\frac{1}{2\pi i} \oint_{\gamma+} \frac{A(z) \exp\left(-h_-(z)\right)}{z-u}\, dz}_{= \text{Residue at } z=\infty} + O\left(\tfrac{1}{N}\right)$$

$A(z) \to 1$, $z \to \infty$, $h_-(z) \to 0$

$= 1$

$$\boxed{A(u) \exp\left(-h_-(u)\right) + 1 + O\left(\tfrac{1}{N}\right) = 0}$$