STAT260 Mean Field Asymptotics in Statistical Learning Lecture 11 - 03/01/2021

Lecture 11: $\mathbb{Z}_2$ synchronization and replica symmetry breaking

Lecturer: Song Mei *Scriber: Yaodong Yu* *Proof reader: Zitong Yang*

This lecture is a continuation of previous lectures on explaining the $\mathbb{Z}_2$ synchronization model. To start with, we will first fix an issue left in the previous lecture in Section 1.

(*Note: This is one of the most difficult lecture in this course. The content of this lecture is not a prerequisite for any lectures afterwards.*)

1 $\mathbb{Z}_2$ synchronization

We first consider the $\mathbb{Z}_2$ synchronization described as follows:

$$\begin{aligned}
\text{Signal:}\quad & \boldsymbol{\theta} \in \mathbb{R}^n, \quad \boldsymbol{\theta}_i \sim_{\text{i.i.d.}} \text{Unif}(\{\pm 1\}), \quad \lambda \geq 0,\\
\text{Observation:}\quad & \boldsymbol{Y} = \frac{\lambda}{n}\boldsymbol{\theta}\boldsymbol{\theta}^\top + \boldsymbol{W} \in \mathbb{R}^{n\times n}, \quad \boldsymbol{W} \sim \text{GOE}(n),\\
\text{Estimator:}\quad & \hat{\boldsymbol{\theta}}(\boldsymbol{Y}) = \langle \boldsymbol{\sigma} \rangle_{\beta,\lambda} \equiv \sum_{\boldsymbol{\sigma}\in\mathbb{Z}_2} \boldsymbol{\sigma}\, \mathbb{P}_{\beta,\lambda}(\boldsymbol{\sigma}),\\
\text{Gibbs measure:}\quad & \mathbb{P}_{\beta,\lambda}(\boldsymbol{\sigma}) \propto \exp\{\beta\langle \boldsymbol{\sigma}, \boldsymbol{Y}\boldsymbol{\sigma}\rangle\}\ \boldsymbol{\nu}_o(\boldsymbol{\sigma}),
\end{aligned}$$

where $\boldsymbol{\nu}_o$ is the prior and is a uniform measure. Also note that when $\beta = \infty$, $\hat{\boldsymbol{\theta}}(\boldsymbol{Y})$ gives the maximum likelihood estimator (MLE); When $\beta = \lambda/2$, $\hat{\boldsymbol{\theta}}(\boldsymbol{Y})$ gives the Bayes estimator.

An issue. There is an issue left in the last lecture: because of the symmetric of this Gibbs measure, when we flip all the $\boldsymbol{\sigma}$, the Gibbs measure stays the same, i.e.,

$$\mathbb{P}_{\beta,\lambda}(\boldsymbol{\sigma}) = \mathbb{P}_{\beta,\lambda}(-\boldsymbol{\sigma}) \Rightarrow \langle \boldsymbol{\sigma} \rangle_{\beta,\lambda} = 0,$$

which makes the estimator not informative for any finite β.

A solution. One solution to the above issue could be: Prior $\boldsymbol{\nu}_\varepsilon$ on the ground truth parameter $\boldsymbol{\theta}_i$, where

$$\boldsymbol{\theta}_i = \begin{cases} 1, & \text{w.p}\ \ \dfrac{1+\varepsilon}{2},\\ -1, & \text{w.p}\ \ \dfrac{1-\varepsilon}{2}, \end{cases}$$

where ε is very small positive parameter. Then we define the tilted Gibbs measure as follows:

$$\mathbb{P}_{\beta,\lambda,\varepsilon}(\boldsymbol{\sigma}) \propto \exp\{\beta\langle \boldsymbol{\sigma}, \boldsymbol{Y}\boldsymbol{\sigma}\rangle\}\ \boldsymbol{\nu}_\varepsilon(\boldsymbol{\sigma}),$$

where ε is the tilted prior on $\boldsymbol{\theta}$.

By considering the tilted Gibbs measure, we can define the **limiting observables** we are interested in,

$$\begin{aligned}
m_*(\beta,\lambda) &\equiv \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \mathbb{E}\left[\langle \frac{1}{n}\sum_{i=1}^n \psi(\boldsymbol{\sigma}_i, \boldsymbol{\theta}_i)\rangle_{\beta,\lambda,\varepsilon}\right],\\
s_*(\beta,\lambda) &\equiv \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \psi(\langle \boldsymbol{\sigma}_i\rangle_{\beta,\lambda,\varepsilon}, \boldsymbol{\theta}_i)\right],
\end{aligned}$$

where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is the sufficiently smooth test function. Note that the difference between $m_*(\beta, \lambda)$ and $s_*(\beta, \lambda)$ is that the ensemble average is take inside the ψ function for $s_*(\beta, \lambda)$. Also, the $s_*(\beta, \lambda)$ is more important than $m_*(\beta, \lambda)$ when we study the properties of the estimator. For now, we will first ignore this ε issue. This does not affect much the *replica calculation* (i.e., keep this ε and take it goes to 0).

2 Formalism

In this section, we will show the formalism that we studied in the previous lecture. (*All the results are in the sense of taking $\mathbb{P}_{\beta,\lambda,\varepsilon}$ and send $\varepsilon \to 0$.*)
We first look at the following $m_*(\beta, \lambda)$ quantity,

$$(\mathbf{A}) \quad m_*(\beta, \lambda) \equiv \lim_{\varepsilon \to 0+} \lim_{n \to \infty} \mathbb{E}\left[\langle \frac{1}{n} \sum_{i=1}^{n} \psi(\boldsymbol{\sigma}_i, \boldsymbol{\theta}_i) \rangle_{\beta,\lambda,\varepsilon} \right] = \mathbb{E}_{G,\theta} \left\{ \mathbb{E}_{\bar{\boldsymbol{\sigma}} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))} \left[\psi(\bar{\boldsymbol{\sigma}}, \theta) \right] \right\},$$

where $G \sim \mathcal{N}(0, 1)$, $\theta \sim \text{Unif}(\{\pm 1\})$. Also we have

$$\bar{\boldsymbol{\sigma}} \in \{\pm 1\}, \quad \mathbb{E}\left[\bar{\boldsymbol{\sigma}}\right] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G).$$

where q_*, μ_* are two parameters and can be described as solutions of the following self-consistent equations:

$$\begin{aligned} q_* &= \mathbb{E}_{G,\theta} \left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)^2 \right], \\ \mu_* &= \mathbb{E}_{G,\theta} \left[\theta \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G) \right], \end{aligned}$$

where $\mu_* > 0$. Actually,

$$\frac{1}{n} \sum_{i=1}^{n} \psi(\boldsymbol{\sigma}_i, \theta_i) \to \mathbb{E}\left[\psi(\bar{\boldsymbol{\sigma}}, \theta)\right], \quad \textit{in probability under} \quad \mathbb{E}\left[\langle \cdot \rangle_{\beta,\lambda}\right].$$

$$\text{Interpolation:} \quad \frac{1}{n} \sum_{i=1}^{n} \delta_{(\boldsymbol{\sigma}_i, \theta_i)} \to \text{Law of } (\bar{\boldsymbol{\sigma}}, \theta),$$

and the joint distribution is defined as follows,

$$\theta \sim \text{Unif}(\{\pm 1\}), \quad G \sim \mathcal{N}(0, 1), \quad \bar{\boldsymbol{\sigma}} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)),$$

and this can be interpreted as: we first sample $\theta \sim \text{Unif}(\{\pm 1\})$, then we independent Gaussian random variable $G \sim \mathcal{N}(0, 1)$, and we get two random variables (θ, G). Then we sample another random variable $\bar{\boldsymbol{\sigma}}$ which depends on (θ, G). The distribution D can be described in terms of a single variable $\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)$, i.e., the distribution D is a binary $\{\pm\}$ and its mean is $\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)$. Note that we have derived the above results in the previous lecture.

Next, we will study the $s_*(\beta, \lambda)$ in this lecture,

$$(\mathbf{B}) \quad s_*(\beta, \lambda) \equiv \lim_{\varepsilon \to 0+} \lim_{n \to \infty} \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \psi(\langle \boldsymbol{\sigma}_i \rangle_{\beta,\lambda,\varepsilon}, \theta_i) \right] = \mathbb{E}_{G,\theta} \left[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta) \right].$$

The interpolation of this $m_*(\beta, \lambda)$ can be described as follows,

$$\text{Interpolation:} \quad \frac{1}{n} \sum_{i=1}^{n} \delta_{(\langle \boldsymbol{\sigma}_i \rangle_{\beta,\lambda}, \theta_i)} \to \text{Law of } (m, \theta),$$

where

$$\theta \sim \text{Unif}(\{\pm 1\}), \quad G \sim \mathcal{N}(0, 1), \quad m = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G),$$

where m can be regarded as $m = \mathbb{E}\left[\bar{\boldsymbol{\sigma}}\right]$, and the expectation is under the D distribution.

To calculate the $s_*(\beta, \lambda)$, we will first introduce the cheating principle.

A cheating principle:

When replica symmetric ansatz holds, suppose we have $\forall \psi : \mathbb{R}^2 \to \mathbb{R}$, smooth test function,

$$\lim_{n\to\infty} \mathbb{E}\left[\langle \frac{1}{n}\sum_{i=1}^{n} \psi(\boldsymbol{\sigma}_i, \theta_i)\rangle_\beta\right] = \mathbb{E}_{\theta,x}\left[\mathbb{E}_{\bar{\boldsymbol{\sigma}}\sim D(\theta,x)}\left[\psi(\bar{\boldsymbol{\sigma}}, \theta)\right]\right].$$

which implies that $\forall \psi : \mathbb{R}^2 \to \mathbb{R}$, smooth test function,

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} \psi(\langle \boldsymbol{\sigma}_i\rangle_\beta, \theta_i)\right] = \mathbb{E}_{\theta,x}\left[\psi(\mathbb{E}_{\bar{\boldsymbol{\sigma}}\sim D(\theta,x)}\left[\bar{\boldsymbol{\sigma}}\right], \theta)\right].$$

2.1 Free energy trick and replica trick for $m_*(\beta, \lambda)$

In the last lecture, we have shown that how to use the free energy trick and replica trick to derive the $m_*(\beta, \lambda)$ quantity. In this subsection, we will briefly review the contents of the last lecture.

To start with, we assume $\Omega = \mathbb{Z}_2^n$ and $\boldsymbol{\nu}_0 = \text{Unif}$. Then

$$\begin{aligned}
H_{\lambda,h}(\boldsymbol{\sigma}) &= -\langle \boldsymbol{\sigma}, \boldsymbol{W}\boldsymbol{\sigma}\rangle - \lambda\langle \boldsymbol{\sigma}, \theta\rangle^2 - h\sum_{i=1}^{n} \psi(\underbrace{\boldsymbol{\sigma}_i}_{\text{perturbation}}, \theta_i),\\
Z_n(\beta, \lambda, h) &= \int_\Omega \exp\left\{-\beta H_{\lambda,h}(\boldsymbol{\sigma})\right\} \boldsymbol{\nu}_0(d\boldsymbol{\sigma}),\\
\varphi(\beta, \lambda, h) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log Z_n(\beta, \lambda, h)\right],\\
m_*(\beta, \lambda) &= \frac{1}{\beta}\partial_h \varphi(\beta, \lambda, h)\Big|_{h=0}.
\end{aligned}$$

Next, we consider the following three terms (by using the replica trick):

$$\begin{aligned}
&(a). \quad S(k, \beta, \lambda, h) \equiv \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left[Z_n(\beta, \lambda, h)^k\right], \quad \textit{(The n limit.)}\\
&(b). \quad \varphi(\beta, \lambda, h) \equiv \lim_{k\to 0} \frac{1}{k} S(k, \beta, \lambda, h), \quad \textit{(The k limit.)}\\
&(c). \quad m_*(\beta, \lambda) \equiv \frac{1}{\beta}\partial_h \varphi(\beta, \lambda, h)\Big|_{h=0}. \quad \textit{(The h derivative.)}
\end{aligned}$$

2.1.1 (a). The $n \to \infty$ limit

To start with, recall that the rate function of the moment can be expressed (using the variational formula) as follows,

$$S(k, \beta, \lambda, h) = \underset{\boldsymbol{\mu}\in\mathbb{R}^k, \boldsymbol{Q}\in\mathbb{R}^{k\times k}, \boldsymbol{Q}_{ii}=1, \boldsymbol{Q}\succeq 0}{\text{ext}} U(\boldsymbol{\mu}, \boldsymbol{Q}).$$

Suppose $\theta \sim \text{Unif}(\{\pm 1\})$, $\boldsymbol{\sigma} \sim \text{Unif}(\{\pm 1\}^k)$, and the U function can be expressed as:

$$U(\boldsymbol{\mu}, \boldsymbol{Q}) = -\beta\lambda\sum_{a=1}^{k} \boldsymbol{\mu}_a^2 - \beta^2\sum_{ab=1}^{k} \boldsymbol{q}_{ab}^2 + \log \mathbb{E}_{\theta,\boldsymbol{\sigma}}\left[\exp\left\{2\beta\lambda\sum_{a=1}^{k} \boldsymbol{\mu}_a\boldsymbol{\sigma}_a\theta + 2\beta^2\sum_{ab=1}^{k} \boldsymbol{q}_{ab}\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b + \beta h\sum_{a=1}^{k} \psi(\boldsymbol{\sigma}_a, \theta)\right\}\right],$$

where the log expectation term corresponds to the entropy term.

2.1.2 (b). The $k \to 0$ limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0}\frac{1}{k}\left\{\underset{\boldsymbol{\mu}\in\mathbb{R}^k,\boldsymbol{Q}\in\mathbb{R}^{k\times k},\boldsymbol{Q}_{ii}=1,\boldsymbol{Q}\succeq 0}{\text{ext}} U(\boldsymbol{\mu},\boldsymbol{Q})\right\}.$$

Next we study how to simplify the above expression.

Trick 0: Replica symmetric ansatz. (*consider the stationary point.*)

$$\begin{cases} \boldsymbol{\mu}_a = \mu, & 1 \le a \le k, \\ \boldsymbol{q}_{ab} = q, & 1 \le a \ne b \le k, \end{cases}$$

After we plug in the above expression into U, we can get a simplified expression as follows,

$$\begin{aligned} U(\boldsymbol{\mu},\boldsymbol{Q}) =& -\beta\lambda k\mu^2 - \beta^2\left(k + k(k-1)q^2\right) + 2\beta^2(1-q) \\ &+ \log\mathbb{E}_{\boldsymbol{\sigma},\theta}\left[\exp\left\{\beta\lambda\mu\sum_{a=1}^k \boldsymbol{\sigma}_a\theta + 2\beta^2 q\left(\sum_{a=1}^k\boldsymbol{\sigma}_a\right)^2 + \beta h\sum_{a=1}^k\psi(\boldsymbol{\sigma}_a,\theta)\right\}\right], \end{aligned}$$

where we can find that U depends on k more explicitly.

Trick 1: To handle the entropy term, we assume $G \sim \mathcal{N}(0,1)$,

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{\lambda G\sum_{a=1}^k\boldsymbol{\sigma}_a\right\}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_{a=1}^k\boldsymbol{\sigma}_a\right)^2\right\} \\ &= -\beta\lambda k\mu^2 - \beta^2\left(k + k(k-1)q^2\right) + 2\beta^2(1-q)k \\ &\quad + \log\mathbb{E}_{G,\boldsymbol{\sigma},\theta}\left[\exp\left\{2\beta\lambda\mu\sum_{a=1}^k\boldsymbol{\sigma}_a\theta + 2\beta\sqrt{q}G\sum_{a=1}^k\boldsymbol{\sigma}_a + \beta h\sum_{a=1}^k\psi(\boldsymbol{\sigma}_a,\theta)\right\}\right] \\ &= -\beta\lambda k\mu^2 - \beta^2\left(k + k(k-1)q^2\right) + 2\beta^2(1-q)k + \log\mathbb{E}_{G,\theta}\left[\left(\mathbb{E}_{\boldsymbol{\sigma}}\exp\left\{2\beta\lambda\mu\boldsymbol{\sigma}\theta + 2\beta\sqrt{q}G\boldsymbol{\sigma} + \beta h\psi(\boldsymbol{\sigma},\theta)\right\}\right)^k\right]. \end{aligned}$$

By introducing the G variable, we can factorize the exponential term in the entropy term into single exponential to the power of k.

Trick 2: The reverse replica trick: $\lim_{k\to 0}\frac{1}{k}\log\mathbb{E}_x\left[Z(x)^k\right] = \mathbb{E}_x\left[\log Z(x)\right]$.

$$\begin{aligned} &\lim_{k\to 0}\frac{1}{k}U(\boldsymbol{\mu},q) \\ =& -\beta\lambda\mu^2 \underbrace{-\beta^2(1-q^2) + 2\beta^2(1-q)}_{\beta^2(1-q)^2} + \mathbb{E}_{G,\theta}\left\{\log\left[\mathbb{E}_{\boldsymbol{\sigma}}\exp\left\{2\beta\lambda\mu\boldsymbol{\sigma}\theta + 2\beta\sqrt{q}G\boldsymbol{\sigma} + \beta h\psi(\boldsymbol{\sigma},\theta)\right\}\right]\right\} \\ \equiv& \, u(\boldsymbol{\mu},q;\beta,\lambda,h), \end{aligned}$$

where $(G,\beta,\boldsymbol{\sigma}) \sim \mathcal{N}(0,1)\times\text{Unif}(\{\pm1\})\times\text{Unif}(\{\pm1\})$.

$$\varphi(\beta,\lambda,h) = \underset{\boldsymbol{\mu},q}{\text{ext}}\; u(\boldsymbol{\mu},q;\beta,\lambda,h),$$

where

$$u(\boldsymbol{\mu},q;\beta,\lambda,h) = -\beta\lambda\mu^2 + \beta^2(1-q)^2 + \mathbb{E}_{G,\theta}\left\{\log\left[\mathbb{E}_{\boldsymbol{\sigma}}\exp\left\{2\beta\lambda\mu\boldsymbol{\sigma}\theta + 2\beta\sqrt{q}G\boldsymbol{\sigma} + \beta h\psi(\boldsymbol{\sigma},\theta)\right\}\right]\right\}.$$

2.1.3 (c). The h derivative

$$\begin{aligned}\frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0} &= \frac{1}{\beta}\partial_h u(\boldsymbol{\mu},q;\beta,\lambda,h)\Big|_{(q=q_*,\mu=\mu_*,h=0)}\\ &= \mathbb{E}_{G,\theta}\left[\frac{\mathbb{E}_{\boldsymbol{\sigma}}\left[\exp\left\{2\beta\lambda\mu_*\boldsymbol{\sigma}\theta+2\beta\sqrt{q_*G\boldsymbol{\sigma}}\right\}\psi(\boldsymbol{\sigma},\theta)\right]}{\mathbb{E}_{\boldsymbol{\sigma}}\left[\exp\left\{2\beta\lambda\mu_*\boldsymbol{\sigma}\theta+2\beta\sqrt{q_*G\boldsymbol{\sigma}}\right\}\right]}\right]\\ &= \mathbb{E}_{G,\theta}\left[\mathbb{E}_{\bar{\boldsymbol{\sigma}}\sim D}\left[\psi(\bar{\boldsymbol{\sigma}},\theta)\right]\right],\end{aligned}$$

where $\bar{\boldsymbol{\sigma}}$ satisfies

$$\mathbb{E}[\bar{\boldsymbol{\sigma}}]=\tanh(2\beta\lambda\mu_*\theta+2\beta\sqrt{q_*}G),$$

and

$$(\mu_*,q_*)=\operatorname*{arg\,ext}_{\boldsymbol{\mu},\boldsymbol{q}} u(\boldsymbol{\mu},q;\beta,\lambda,h)\Big|_{h=0}.$$

Therefore, we have

$$\begin{cases}\mu_*=\mathbb{E}_{G,\theta}\left[\theta\cdot\tanh\left(2\beta\lambda\mu_*\theta+2\beta\sqrt{q_*G}\right)\right],\\ q_*=\mathbb{E}_{G,\theta}\left[\tanh\left(2\beta\lambda\mu_*\theta+2\beta\sqrt{q_*G}\right)^2\right].\end{cases}$$

2.2 Free energy trick and replica trick for $s_*(\beta,\lambda)$

In this lecture, we will study the following $s_*(\beta,\lambda)$ quantity:

$$\text{Our goal:}\quad s_*(\beta,\lambda)\equiv\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\psi(\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\beta_i)\right].$$

We can find that the $s_*(\beta,\lambda)$ quantity is a bit hard to calculate since it is hard to express $\frac{1}{n}\sum_{i=1}^{n}\psi(\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\beta_i)$ in terms of ensemble average of some Gibbs distributions. To start with, we first look at the observation. Observation: Let $(\boldsymbol{\sigma}^1,\boldsymbol{\sigma}^2,\cdots,\boldsymbol{\sigma}^N)\sim\mathbb{P}_{\beta,\lambda}^{\otimes N}$, where N is the number of replicas. Then if we fix i,

$$\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a\underset{N\to\infty}{\rightarrow}\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\quad\textit{(Law of large number).}$$

Then we look at the following quantity,

$$\frac{1}{n}\sum_{i=1}^{n}\psi\left(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a,\theta_i\right)\underset{N\to\infty}{\rightarrow}\frac{1}{n}\sum_{i=1}^{n}\psi\left(\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\theta_i\right).\tag{1}$$

Next we can define the ensemble average of a function of these explicit replicas, i.e.,

$$\langle f(\boldsymbol{\sigma}^1,\cdots,\boldsymbol{\sigma}^N)\rangle_{\beta,\lambda,N}=\int_{\Omega^{\otimes N}}f(\boldsymbol{\sigma}^1,\cdots,\boldsymbol{\sigma}^N)\prod_{a=1}^{N}\mathbb{P}_{\beta,\lambda}(d\boldsymbol{\sigma}).$$

Then our are going to calculate the following term

$$\begin{aligned}s_*(\beta,\lambda)&\equiv\lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\psi\left(\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\theta_i\right)\right]\\ &=\lim_{n\to\infty}\lim_{N\to\infty}\mathbb{E}\left[\langle\frac{1}{n}\sum_{i=1}^{n}\psi\left(\boldsymbol{\sigma}_i,\theta_i\right)\rangle_{\beta,\lambda}\right]\\ &\overset{?}{=}\lim_{N\to\infty}\lim_{n\to\infty}\mathbb{E}\left[\cdots\right],\end{aligned}$$

where the second equality is because of Eq. (1), and the question is whether we could exchange the limits. We could use the free energy trick to calculate the inner limit (*limiting observable in the finite replica case*), and then send the replica to infinity. Next we first introduce the free energy trick.

Free energy trick:

$$\underline{\Omega} = \Omega^{\otimes N}, \quad \underline{\boldsymbol{\nu}_0} = \boldsymbol{\nu}_0^{\otimes N}, \quad \underline{\boldsymbol{\sigma}} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2, \cdots, \underbrace{\boldsymbol{\sigma}^N}_{\text{“E-replica”}}).$$

Then we define the following terms,

$$\begin{aligned} H_{\lambda,h,N}(\underline{\boldsymbol{\sigma}}) &= \sum_{a=1}^{N} H_\lambda(\boldsymbol{\sigma}^a) - h\sum_{i=1}^{n} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a, \theta_i\right), \\ \varphi(\beta,\lambda,h,N) &= \lim_{n\to\infty}\frac{1}{n}\log\int_{\underline{\Omega}}\exp\left\{-\beta H_{\lambda,h,N}(\underline{\boldsymbol{\sigma}})\right\}\underline{\boldsymbol{\nu}_0}, \\ \partial_h\varphi(\beta,\lambda,0,N) &= \lim_{n\to\infty}\left\{\mathbb{E}\left[\langle\sum_{i=1}^{n}\psi(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a,\theta_i)\rangle_{\beta,\lambda,N}\right]\Big/ n\right\}. \end{aligned}$$

We expect (*which is our plan*):

$$s_*(\beta,\lambda) \equiv \lim_{n\to\infty}\mathbb{E}\left[\sum_{i=1}^{n}\psi(\langle\boldsymbol{\sigma}_i\rangle_{\beta,\lambda},\theta_i)\right]\Big/ n = \lim_{N\to\infty}\partial_h\varphi(\beta,\lambda,0,N).$$

In fact, we will show that:

$$\begin{aligned} &\lim_{n\to\infty}\mathbb{E}\left[\langle\frac{1}{n}\sum_{i=1}^{n}\psi\Big(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a,\theta_i\Big)\rangle_{\beta,\lambda}\right] \\ =&\,\partial_h\varphi(\beta,\lambda,0,N) \\ =&\,\mathbb{E}_{G,\theta}\left\{\mathbb{E}_{\bar{\boldsymbol{\sigma}}^a\sim_{\text{i.i.d.}}D(\tanh(2\beta\lambda\mu_*\theta+2\beta\sqrt{q_*}G))}\left[\psi(\frac{1}{N}\sum_{a=1}^{N}\bar{\boldsymbol{\sigma}}^a,\theta)\right]\right\}, \end{aligned}$$

therefore,

$$\lim_{N\to\infty}\partial_h\varphi(\beta,\lambda,0,N) = \mathbb{E}_{G,\theta}\left[\psi(\tanh(2\beta\lambda\mu_*\theta+2\beta\sqrt{q_*}G),\theta)\right].$$

Next we present the Replica tricks:

$$(a).\quad S(k,\beta,\lambda,h,N) = \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left(\int_{\underline{\Omega}}\exp\left\{-\beta H_{\lambda,h,N}(\underline{\boldsymbol{\sigma}})\right\}\underline{\boldsymbol{\nu}_0}d(\underline{\boldsymbol{\sigma}})\right)^k, \quad \textit{(The n limit.)}$$

$$(b).\quad \varphi(\beta,\lambda,h,N) = \lim_{k\to 0}\frac{1}{k}S(k,\beta,\lambda,h,N), \quad \textit{(The k limit.)}$$

$$(c).\quad s_*(\beta,\lambda) = \lim_{N\to\infty}\partial_h\varphi(\beta,\lambda,h,N). \quad \textit{(The h derivatives.)}$$

2.2.1 (a). The n limit

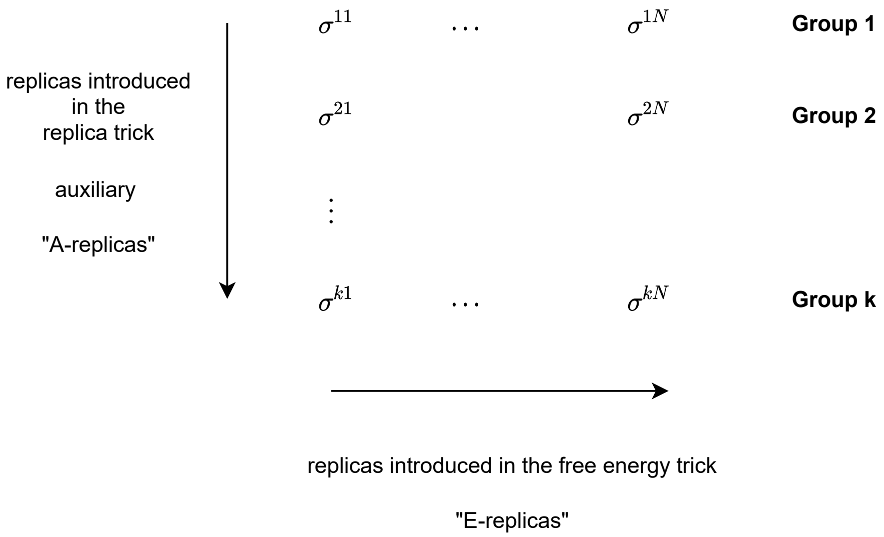
First of all, we have

$$H_{\lambda,h,N}(\underline{\boldsymbol{\sigma}}) = \sum_{a=1}^{N} H_\lambda(\boldsymbol{\sigma}^a) - h\sum_{i=1}^{n} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^a, \theta_i\right), \tag{2}$$

then we compute $\mathbb{E}[Z_n^k]$ (*by expanding the integrals*):

$$\mathbb{E}[Z_n^k] = \mathbb{E}\left[\int_{\Omega^{\otimes Nk}} \exp\left\{-\beta\sum_{b=1}^{k}\left(\sum_{a=1}^{N} H_{\beta,\lambda}(\boldsymbol{\sigma}^{ab}) + \beta h\sum_{i=1}^{n}\psi(\sum_{a=1}^{N}\boldsymbol{\sigma}_i^{ab},\theta_i)\right)\right\}\prod_{a=1}^{N}\prod_{b=1}^{k}\boldsymbol{\nu}_0(d\boldsymbol{\sigma}^{ab})\right]$$

There are two types of replicas: explicit replicas ("*E-replicas*") and auxiliary replicas ("*A-replicas*"):



We can find that there are some symmetries and some breaking symmetries in the above expression Eq. (2):

1. If $h = 0$: all the replicas are symmetric.
2. If $h \neq 0$: symmetric is broken.

Also, note that

- Invariant if exchange replicas in the same group. ✓
- Invariant if exchange groups. ✓
- Non-invariant if exchange replicas in different groups. ✗

Next we present the calculation of $S(k,\beta,\lambda,h,N)$. To start with, we have

$$S(k,\beta,\lambda,h,N) = \underset{\boldsymbol{\mu}\in\mathbb{R}^k,\boldsymbol{Q}\in\mathbb{R}^{Nk\times Nk},\boldsymbol{Q}_{ii}=1,\boldsymbol{Q}\succeq 0}{\mathrm{ext}} U(\boldsymbol{\mu},\boldsymbol{Q}),$$

where $U(\boldsymbol{\mu},\boldsymbol{Q})$ is defined as

$$\begin{aligned} U(\boldsymbol{\mu},\boldsymbol{Q}) = &-\beta\lambda\sum_{b=1}^{N}\sum_{a=1}^{k}\boldsymbol{\mu}_{ab}^2 - \beta^2\sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k}\boldsymbol{q}_{ab,a'b'}^2 \\ &+\log\mathbb{E}_{\boldsymbol{\sigma},\theta}\left[\exp\left\{2\beta\lambda\sum_{b=1}^{k}\sum_{a=1}^{N}\boldsymbol{\mu}_{ab}\boldsymbol{\sigma}^{ab}\theta + 2\beta^2\sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k}\boldsymbol{q}_{ab,a'b'}\boldsymbol{\sigma}^{ab}\boldsymbol{\sigma}^{a'b'} + \beta h\sum_{b=1}^{k}\psi(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}_i^{ab},\theta)\right\}\right]. \end{aligned} \tag{3}$$

where $\theta \sim \text{Unif}(\{\pm 1\})$, $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^{ab})_{1\le a\le N, 1\le k\le K} \sim_{\text{i.i.d.}} \text{Unif}(\{\pm 1\})$, and

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{q}_{11,11} & \cdots & \cdots & \boldsymbol{q}_{11,Nk} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{q}_{Nk,11} & \cdots & \cdots & \boldsymbol{q}_{Nk,Nk} \end{bmatrix}, \quad \boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_{11} & \cdots & \boldsymbol{\mu}_{Nk} \end{bmatrix}.$$

The consequence will happen in the small k limit.

2.2.2 (b). The $k \to 0$ limit

Replica symmetric ansatz ($k = 3, N = 2$), $\boldsymbol{Q}$ is in block structure, i.e., where q_1 may not equal to q_0, also note that we can only change the index of $\boldsymbol{Q}$ which are invariant, and $\boldsymbol{Q}$ is deinfed as follows:

$$\begin{array}{c}\left[\begin{array}{cc:cc:cc} 1 & q_1 & q_0 & q_0 & q_0 & q_0 \\ q_1 & 1 & q_0 & q_0 & q_0 & q_0 \\ \hdashline q_0 & q_0 & 1 & q_1 & q_0 & q_0 \\ q_0 & q_0 & q_1 & 1 & q_0 & q_0 \\ \hdashline q_0 & q_0 & q_0 & q_0 & 1 & q_1 \\ q_0 & q_0 & q_0 & q_0 & q_1 & 1 \end{array}\right]\begin{array}{l} \left.\begin{array}{c} \\ \\ \end{array}\right] \text{Group 1} \\ \left.\begin{array}{c} \\ \\ \end{array}\right] \text{Group 2} \\ \left.\begin{array}{c} \\ \\ \end{array}\right] \text{Group 3} \end{array} \\ \begin{array}{ccc} \underbrace{\qquad\quad}_{\text{Group 1}} & \underbrace{\qquad\quad}_{\text{Group 2}} & \underbrace{\qquad\quad}_{\text{Group 3}} \end{array} \end{array}$$

We define $\boldsymbol{\mu}$ as $\boldsymbol{\mu} = (\mu, \mu, \cdots, \mu)$.

Claim: *There must be a stationary point of U in Eq. (3) that satisfies the above form.*

You could check the above claim by looking at the symmetry of $U(\boldsymbol{\mu}, \boldsymbol{Q})$.

Then we plug in the $\boldsymbol{Q}$ and $\boldsymbol{\mu}$ into the function $U(\boldsymbol{\mu}, \boldsymbol{Q})$, and compute

$$U(k, \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{\mu}, N) = -\beta\lambda Nk\boldsymbol{\mu}^2 - \beta^2\left[Nk(1-\boldsymbol{q}_1)^2 + N^2k(\boldsymbol{q}_1-\boldsymbol{q}_0)^2 + N^2k^2\boldsymbol{q}_0^2 - 2(1-\boldsymbol{q}_1)Nk\right] + T,$$

where the quadratic parts are similar, and the entropy term is

$$T \equiv \log \mathbb{E}_{\boldsymbol{\sigma},\theta}\left[\exp\left\{2\beta\lambda\boldsymbol{\mu}\sum_{ab}\boldsymbol{\sigma}^{ab}\theta + 2\beta^2(\boldsymbol{q}_1-\boldsymbol{q}_0)\sum_{b=1}^{k}(\sum_{a=1}^{N}\boldsymbol{\sigma}^{ab})^2 + 2\beta^2\boldsymbol{q}_0(\sum_{b=1}^{k}\sum_{a=1}^{N}\boldsymbol{\sigma}^{ab})^2 + \beta h\sum_{b=1}^{k}\psi(\frac{1}{N}\sum_{a=1}^{N}\boldsymbol{\sigma}^{ab}, \theta)\right\}\right],$$

also by introducing the Gaussian variable G_0,

$$\mathbb{E}\left[\exp\left\{\lambda G_0 \sum_b \sum_a \boldsymbol{\sigma}^{ab}\right\}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_b\sum_a \boldsymbol{\sigma}^{ab}\right)^2\right\}.$$

Then the calculation tricks are similar as before, we can calculate T as follows

$$\begin{aligned}
T &= \log \mathbb{E}_{\boldsymbol{\sigma}, G_0, \theta} \left[\exp \left\{ 2\beta\lambda\boldsymbol{\mu} \sum_{ab} \boldsymbol{\sigma}^{ab}\theta + 2\beta^2(\boldsymbol{q}_1 - \boldsymbol{q}_0) \sum_{b=1}^{k} (\sum_{a=1}^{N} \boldsymbol{\sigma}^{ab})^2 + 2\beta\sqrt{\boldsymbol{q}_0} G_0 \sum_{b=1}^{k}\sum_{a=1}^{N} \boldsymbol{\sigma}^{ab} + \beta h \sum_{b=1}^{k} \psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^{ab}, \theta) \right\} \right] \\
&= \log \mathbb{E}_{\boldsymbol{\sigma}, G_0} \left[\left(\mathbb{E}_{\boldsymbol{\sigma}} \exp \left\{ 2\beta\lambda\boldsymbol{\mu} \sum_{a} \boldsymbol{\sigma}^{a}\theta + 2\beta^2(\boldsymbol{q}_1 - \boldsymbol{q}_0)(\sum_{a=1}^{N} \boldsymbol{\sigma}^{a})^2 + 2\beta\sqrt{\boldsymbol{q}_0} G_0 \sum_{a=1}^{N} \boldsymbol{\sigma}^{a} + \beta h \psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^{a}, \theta) \right\} \right)^k \right] \\
&= \log \mathbb{E}_{\boldsymbol{\sigma}, G_0} \left[\left(\mathbb{E}_{\boldsymbol{\sigma}, G_1} \exp \left\{ 2\beta\lambda\boldsymbol{\mu} \sum_{a} \boldsymbol{\sigma}^{a}\theta + 2\beta\sqrt{\boldsymbol{q}_1 - \boldsymbol{q}_0} G_1 \sum_{a=1}^{N} \boldsymbol{\sigma}^{a} + 2\beta\sqrt{\boldsymbol{q}_0} G_0 \sum_{a=1}^{N} \boldsymbol{\sigma}^{a} + \beta h \psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^{a}, \theta) \right\} \right)^k \right],
\end{aligned}$$

where we eliminate b in the second equality, and we introduce another Gaussian variable in the last step. Now we have a simplified expression and the free entropy density can be calculated as

$$\begin{aligned}
&\varphi(\beta, \lambda, h, N) \\
\equiv &\lim_{k\to 0} \sup_{\boldsymbol{q}_1, \boldsymbol{q}_0, \boldsymbol{\mu}} \frac{1}{k} U(k, \boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{\mu}, N) \\
= &-\beta\lambda N - \beta 2 \left[N(1-\boldsymbol{q}_1)^2 + N^2(\boldsymbol{q}_1 - \boldsymbol{q}_0)^2 - 2(1-\boldsymbol{q}_1)N \right] \\
&+ \mathbb{E}_{\theta, G_0} \left[\log \mathbb{E}_{\boldsymbol{\sigma}, G_1} \left[\exp \left\{ 2\beta\lambda\boldsymbol{\mu} \sum_{a} \boldsymbol{\sigma}^{a}\theta + 2\beta\sqrt{\boldsymbol{q}_1 - \boldsymbol{q}_0} G_1 \sum_{a=1}^{N} \boldsymbol{\sigma}^{a} + 2\beta\sqrt{\boldsymbol{q}_0} G_0 \sum_{a=1}^{N} \boldsymbol{\sigma}^{a} + \beta h \psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^{a}, \theta) \right\} \right] \right],
\end{aligned}$$

where we have an additional term $2\beta\sqrt{\boldsymbol{q}_0} G_0 \sum_{a=1}^{N} \boldsymbol{\sigma}^a$ compared with previous calculations.

2.2.3 (c). The h derivatives

By applying the implicit differentiation theorem, we can calculate $\partial_h \varphi(\beta, \lambda, h, N)$ as follows

$$\begin{aligned}
&\partial_h \varphi(\beta, \lambda, h, N) \\
&= \partial_h u(\boldsymbol{q}_1, \boldsymbol{q}_0, \boldsymbol{\mu}; \beta, \lambda, 0, N) \\
&= \mathbb{E}_{\theta, G_0} \left[\frac{\mathbb{E}_{\boldsymbol{\sigma}, G_1} \left[\exp \left\{ 2\beta\lambda\boldsymbol{\mu}_* \sum_a \boldsymbol{\sigma}^a\theta + 2\beta(\sqrt{\boldsymbol{q}_{1*} - \boldsymbol{q}_{0*}} G_1 + \sqrt{\boldsymbol{q}_{0*}} G_0) \sum_{a=1}^{N} \boldsymbol{\sigma}^a \right\} \psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^a, \theta) \right]}{\mathbb{E}_{\boldsymbol{\sigma}, G_1} \left[\exp \left\{ 2\beta\lambda\boldsymbol{\mu}_* \sum_a \boldsymbol{\sigma}^a\theta + 2\beta(\sqrt{\boldsymbol{q}_{1*} - \boldsymbol{q}_{0*}} G_1 + \sqrt{\boldsymbol{q}_{0*}} G_0) \sum_{a=1}^{N} \boldsymbol{\sigma}^a \right\} \right]} \right],
\end{aligned} \tag{4}$$

where

$$\boldsymbol{\mu}_*,\ \boldsymbol{q}_{1*},\ \boldsymbol{q}_{0*} = \operatorname*{arg\,ext}_{\boldsymbol{\mu}, \boldsymbol{q}_1, \boldsymbol{q}_0} u(\boldsymbol{q}_1, \boldsymbol{q}_0, \boldsymbol{\mu}; \beta, \lambda, 0, N).$$

"*Obviously*", there exists a *stationary point* such that (*when h is 0*)

$$\boldsymbol{q}_{1*}, = \boldsymbol{q}_{0*} = \boldsymbol{q}_*, \quad \text{and} \quad (\boldsymbol{\mu}_*, \boldsymbol{q}_*) = \operatorname*{arg\,ext}_{\boldsymbol{\mu}, \boldsymbol{q}} \underbrace{u(\boldsymbol{q}, \boldsymbol{\mu}; \beta, \lambda, 0)}_{\text{the } N=1 \text{ formula}}.$$

This is the correct stationary point for some region (β, λ). (*Remark: the replica symmetric phase.*) Then we can futher simplify the expression in Eq. (4) as follows,

$$\partial_h \varphi(\beta, \lambda, h, N) = \mathbb{E}_{G, \theta} \left\{ \mathbb{E}_{\bar{\boldsymbol{\sigma}}^a \sim_{\text{i.i.d.}} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))} \left[\psi(\frac{1}{N}\sum_{a=1}^{N} \boldsymbol{\sigma}^a, \theta) \right] \right\}.$$

Therefore, we can calculate $s_*(\beta, \lambda)$:

$$s_*(\beta, \lambda) = \lim_{N\to\infty} \partial_h \varphi(\beta, \lambda, 0, N) = \mathbb{E}_{\theta, G} \left[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta) \right].$$

2.3 1-RSB prediction of $\mathbb{Z}_2$ sync free entropy density

Next, we will have a bried introduction of replica symmetric breaking ansatz. For most models we encounter in this course, we will not use the replica symmetric breaking ansatz. The complication mainly comes from the formula.

To start with, we consider the $\mathbb{Z}_2$ synchronization, we assume $\Omega = \mathbb{Z}_2^n$ and $\boldsymbol{\nu}_0 = \text{Unif}$ (*without considering the perturbation*). Then

$$\begin{aligned} H_{\lambda,h}(\boldsymbol{\sigma}) &= -\langle \boldsymbol{\sigma}, \boldsymbol{W}\boldsymbol{\sigma}\rangle - \lambda\langle \boldsymbol{\sigma}, \theta\rangle^2/n, \\ Z_n(\beta,\lambda,h) &= \int_\Omega \exp\left\{-\beta H_{\lambda,h}(\boldsymbol{\sigma})\right\} \boldsymbol{\nu}_0(d\boldsymbol{\sigma}), \\ \varphi(\beta,\lambda) &= \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\log Z_n(\beta,\lambda)\right]. \end{aligned}$$

We consider the following two terms

$$\begin{aligned} &(a). \quad S(k,\beta,\lambda,h) = \lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[Z_n(\beta,\lambda,h)^k\right], \quad \textit{(The n limit.)} \\ &(b). \quad \varphi(\beta,\lambda,h) = \lim_{k\to 0}\frac{1}{k}S(k,\beta,\lambda,h), \quad \textit{(The k limit.)} \end{aligned}$$

2.3.1 (a). The $n \to \infty$ limit

The calculation of the $n \to \infty$ limit is similar as before,

$$S(k,\beta,\lambda,h) = \underset{\boldsymbol{\mu}\in\mathbb{R}^k, \boldsymbol{Q}\in\mathbb{R}^{k\times k}, \boldsymbol{Q}_{ii}=1, \boldsymbol{Q}\succeq 0}{\text{ext}} U(\boldsymbol{\mu},\boldsymbol{Q}),$$

where $\theta \sim \text{Unif}(\{\pm 1\})$, $\boldsymbol{\sigma} \sim \text{Unif}(\{\pm 1\}^k)$, and

$$U(\boldsymbol{\mu},\boldsymbol{Q}) = -\beta\lambda\sum_{a=1}^k \boldsymbol{\mu}_a^2 - \beta^2\sum_{ab=1}^k \boldsymbol{q}_{ab}^2 + \log\mathbb{E}_{\theta,\boldsymbol{\sigma}}\left[\exp\left\{2\beta\lambda\sum_{a=1}^k \boldsymbol{\mu}_a\boldsymbol{\sigma}_a\theta + 2\beta^2\sum_{ab=1}^k \boldsymbol{q}_{ab}\boldsymbol{\sigma}_a\boldsymbol{\sigma}_b\right\}\right].$$

2.3.2 (b). The $k \to 0$ limit

For the $k \to 0$ limit case, the calculation changes compared with previous ones,

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0}\frac{1}{k}\underset{\boldsymbol{\mu}, \boldsymbol{Q}_{ii}=1, \boldsymbol{Q}\succeq 0}{\text{ext}} U(\boldsymbol{\mu},\boldsymbol{Q}).$$

Replica symmetric ansatz does not always hold. Here we assume 1-step replica symmetric breaking ansatz. Then we compute

$$U(k,\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{\mu},N) = -\beta\lambda k\boldsymbol{\mu}^2 - \beta^2\left[k(k-k_0)\boldsymbol{q}_0^2 + k(k_0-1)\boldsymbol{q}_1^2 + k - 2(1-\boldsymbol{q}_1)k\right] + T,$$

note that there is no perturbation that breaks the symmetry. Then T can be calculated as (*by pluging in*

$$\boldsymbol{Q} = \begin{array}{c}\text{Size } k_1\end{array}\left[\begin{array}{cc:cc:cc} 1 & q_1 & q_0 & q_0 & q_0 & q_0 \\ q_1 & 1 & q_0 & q_0 & q_0 & q_0 \\ \hdashline q_0 & q_0 & 1 & q_1 & q_0 & q_0 \\ q_0 & q_0 & q_1 & 1 & q_0 & q_0 \\ \hdashline q_0 & q_0 & q_0 & q_0 & 1 & q_1 \\ q_0 & q_0 & q_0 & q_0 & q_1 & 1 \end{array}\right]\begin{array}{l}\text{Group 1} \\ \\ \text{Group 2} \\ \\ \\ \\ \end{array}$$

$$\begin{array}{ccc}\text{Group 1} & \text{Group 2} & \text{Group } k_0\end{array}$$

the $\boldsymbol{Q}$ *and* $\boldsymbol{\mu}$):

$$\begin{aligned} T &\equiv \log \mathbb{E}_{\boldsymbol{\sigma},\theta}\left[\exp\left\{2\beta\lambda\boldsymbol{\mu}\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab}\theta + 2\beta^2\boldsymbol{q}_0(\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab})^2 + 2\beta^2(\boldsymbol{q}_1-\boldsymbol{q}_0)\sum_{b=1}^{k_1}(\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab})^2\right\}\right] \\ &= \log \mathbb{E}_{\boldsymbol{\sigma},\theta,G_0}\left[\exp\left\{2\beta\lambda\boldsymbol{\mu}\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0(\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab}) + 2\beta^2(\boldsymbol{q}_1-\boldsymbol{q}_0)\sum_{b=1}^{k_1}(\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab})^2\right\}\right] \\ &= \log \mathbb{E}_{\theta,G_0}\left[\left(\mathbb{E}_{\boldsymbol{\sigma}}\left[\exp\left\{2\beta\lambda\boldsymbol{\mu}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{a}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{a} + 2\beta^2(\boldsymbol{q}_1-\boldsymbol{q}_0)G_1(\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{a})^2\right\}\right]\right)^{k_1}\right] \\ &= \log \mathbb{E}_{\theta,G_0}\left[\left(\mathbb{E}_{\boldsymbol{\sigma},G_1}\left[\exp\left\{2\beta\lambda\boldsymbol{\mu}\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{a}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{a} + 2\beta\sqrt{\boldsymbol{q}_1-\boldsymbol{q}_0}G_1\sum_{a=1}^{k_0}\boldsymbol{\sigma}^{ab}\right\}\right]\right)^{k_1}\right] \\ &= \log \mathbb{E}_{\theta,G_0}\left[\left(\mathbb{E}_{G_1}\left[\mathbb{E}_{\boldsymbol{\sigma}}\left(\exp\left\{2\beta\lambda\boldsymbol{\mu}\boldsymbol{\sigma}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0\boldsymbol{\sigma} + 2\beta\sqrt{\boldsymbol{q}_1-\boldsymbol{q}_0}G_1\boldsymbol{\sigma}\right\}\right)^{k_0}\right]\right)^{k_1}\right] \\ &= \log \mathbb{E}_{\theta,G_0}\left[\left(\mathbb{E}_{G_1}\left[\cosh\left(2\beta\lambda\boldsymbol{\mu}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0 + 2\beta\sqrt{\boldsymbol{q}_1-\boldsymbol{q}_0}G_1\right)^{k_0}\right]\right)^{k/k_0}\right]. \end{aligned}$$

Then we take $k \to 0$:

$$\begin{aligned} &\lim_{k\to 0}\frac{1}{k}U(\boldsymbol{\mu},\boldsymbol{q}_0,\boldsymbol{q}_1,k_0,k) \equiv u(\boldsymbol{\mu},\boldsymbol{q}_0,\boldsymbol{q}_1,k_0,\beta,\lambda) \\ =& -\beta\lambda\boldsymbol{\mu}^2 - \beta^2\left[-k_0\boldsymbol{q}_0^2 + (k_0-1)\boldsymbol{q}_1^2 - 1 - 2(1-\boldsymbol{q}_1)\right] \\ &+ \frac{1}{k_0}\mathbb{E}_{\theta,G_0}\left[\log \mathbb{E}_{G_1}\left[\cosh\left(2\beta\lambda\boldsymbol{\mu}\theta + 2\beta\sqrt{\boldsymbol{q}_0}G_0 + 2\beta\sqrt{\boldsymbol{q}_1-\boldsymbol{q}_0}G_1\right)^{k_0}\right]\right], \end{aligned}$$

therefore,

$$\varphi(\beta,\lambda) = \operatorname*{ext}_{\boldsymbol{\mu},\boldsymbol{q}_0,\boldsymbol{q}_1,k_0} u(\boldsymbol{\mu},\boldsymbol{q}_0,\boldsymbol{q}_1,k_0;\beta,\lambda). \quad (\textit{range of } k_0 \in [0,1].)$$

Remark: To derive the results for the MLE estimator, we need to use a more complicated ansatz, i.e., 2-RSB ansatz:

$$Q = \left[\begin{array}{c:c}
\begin{array}{c:c} \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix} & q_2 \\ \hdashline q_2 & \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix} \end{array} & q_3 \\ \hdashline
q_3 & \begin{array}{c:c} \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix} & q_2 \\ \hdashline q_2 & \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix} \end{array}
\end{array}\right]$$

The ground state of SK model when $\lambda = 0$ is ∞-RSB.