Mean Field Asymptotics in Statistical Learning. Mar 1

Lecture 11. $\mathbb{Z}_2$ synchronization and replica symmetry breaking.

① $\mathbb{Z}_2$ synchronization.

Signal: $\theta \in \mathbb{R}^n$, $\theta_i \sim_{i.i.d.} \mathrm{Unif}(\{\pm 1\})$, $\lambda \geq 0$.

Observation: $Y = \frac{\lambda}{n}\theta\theta^T + W \in \mathbb{R}^{n\times n}$. $W \sim \mathrm{GOE}(n)$.

Estimator: $\hat{\theta}(Y) = \langle \sigma \rangle_{\beta,\lambda} \equiv \sum_{\sigma \in \mathbb{Z}_2} \sigma P_{\beta,\lambda}(\sigma)$.

Gibbs measure: $P_{\beta,\lambda}(\sigma) \propto \exp\{\beta \langle \sigma, Y\sigma \rangle\} \nu_0(\sigma)$

$$\begin{cases} \text{MLE} & \beta = \infty \\ \text{Bayes} & \beta = \lambda/2 \end{cases}$$

An issue: $P_{\beta,\lambda}(\sigma) = P_{\beta,\lambda}(-\sigma)$, $\Rightarrow$ $\langle \sigma \rangle_{\beta,\lambda} = 0$.

A solution: Prior ν_ε on θ_i

$$\theta_i = \begin{cases} 1, & \text{w.p. } \frac{1+\varepsilon}{2}. \\ -1, & \text{w.p. } \frac{1-\varepsilon}{2}. \end{cases}$$

ε is very small.

$$P_{\beta,\lambda,\varepsilon}(\sigma) \propto \exp\{\beta \langle \sigma, Y\sigma \rangle\} \nu_\varepsilon(\sigma)$$

Limiting observables

$$m_*(\beta,\lambda) \equiv \lim_{\varepsilon \to 0+} \lim_{n\to\infty} \mathbb{E}\left[\left\langle \frac{1}{n}\sum_{i=1}^n \psi(\sigma_i, \theta_i)\right\rangle_{\beta,\lambda,\varepsilon}\right].$$

$$s_*(\beta,\lambda) \equiv \lim_{\varepsilon \to 0+} \lim_{n\to\infty} \mathbb{E}\left[\left\langle \frac{1}{n}\sum_{i=1}^n \psi(\langle \sigma_i \rangle_{\beta,\lambda,\varepsilon}, \theta_i)\right\rangle\right].$$

Ignore this ε issue for now.

This doesn't affect much the replica calculation.

Formalism (all the results are in the sense of taking $P_{\beta,\lambda,\varepsilon}$ and then send $\varepsilon \to 0$).

(A) $m_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\left[\left\langle \sum_{i=1}^{n} \psi(\sigma_i, \theta_i) \right\rangle_{\beta,\lambda}\right] / n$

$$= \mathbb{E}_{\underset{\uparrow}{G},\underset{\uparrow}{\theta}} \left\{ \mathbb{E}_{\bar{\sigma} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))} \left[\psi(\bar{\sigma}, \theta)\right]\right\} \quad \left(\begin{array}{l} \bar{\sigma} \in \{\pm 1\} \\ \mathbb{E}[\bar{\sigma}] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G) \end{array} \right)$$

$N(0,1)$ Unif($\{\pm1\}$)

$$q_* = \mathbb{E}_{G,\theta}\left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)^2\right]$$

$$\mu_* = \mathbb{E}_{G,\theta}\left[\theta \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)\right]. \qquad \mu_* > 0.$$

Actually: $\frac{1}{n}\sum_{i=1}^{n} \psi(\sigma_i, \theta_i) \longrightarrow \mathbb{E}[\psi(\bar{\sigma},\theta)]$ in probability under $\mathbb{E}[\langle \cdot \rangle_{\beta,\lambda}]$

Interpretation: $\frac{1}{n}\sum_{i=1}^{n} \delta_{(\sigma_i,\theta_i)} \longrightarrow$ Law of $(\bar{\sigma}, \theta)$

$\theta \sim \mathrm{Unif}(\{\pm1\})$, $G \sim N(0,1)$ $\boxed{\bar{\sigma}} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))$

(B) $s_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\left[\psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\right]/n$

$$= \mathbb{E}_{G,\theta}\left[\psi\left(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta\right)\right].$$

Need more work to show this.

Interpretation: $\frac{1}{n}\sum_{i=1}^{n} \delta_{(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)} \rightarrow$ Law of (m,θ)

$\theta \sim \mathrm{Unif}(\{\pm1\})$, $G \sim N(0,1)$, $\boxed{m} = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)$

$m = \mathbb{E}[\bar{\sigma}]$.

A cheating principle:

When replica symmetric ansatz holds, suppose we have

$\forall \psi: \mathbb{R}^2 \to \mathbb{R}$, smooth test function

$$\lim_{N\to\infty} \mathbb{E}\left[\left\langle \frac{1}{n}\sum_{i=1}^{n}\psi(\sigma_i,\theta_i)\right\rangle_\beta\right] = \mathbb{E}_{\theta,x}\left[\mathbb{E}_{\bar{\sigma}\sim D(\theta,x)}[\psi(\bar{\sigma},\theta)]\right]$$

$\Rightarrow$ $\forall \psi: \mathbb{R}^2 \to \mathbb{R}$, smooth test function

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}\psi(\langle\sigma_i\rangle_\beta, \theta_i)\right] = \mathbb{E}_{\theta,x}\left[\psi(\mathbb{E}_{\bar{\sigma}\sim D(\theta,x)}[\bar{\sigma}], \theta)\right].$$

② Free energy trick and replica trick for $m_*(\beta,\lambda)$.

A quick review of the last lecture.

$$\Omega = \mathbb{Z}_2^n, \quad \nu_0 = \mathrm{Unif}$$

perturbation.

$$H_{\lambda,h}(\sigma) = -\langle \sigma, W\sigma \rangle - \lambda \langle \sigma, \theta \rangle^2/n - h\sum_{i=1}^n \psi(\sigma_i, \theta_i).$$

$$Z_n(\beta,\lambda,h) = \int_\Omega \exp\{-\beta H_{\lambda,h}(\sigma)\}\, \nu_0(d\sigma).$$

$$\varphi(\beta,\lambda,h) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\log Z_n(\beta,\lambda,h)].$$

$$m_*(\beta,\lambda) = \frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}.$$

a) $S(k,\beta,\lambda,h) \equiv \lim_{n\to\infty} \frac{1}{n}\log \mathbb{E}[Z_n(\beta,\lambda,h)^k]$. The n limit.

b) $\varphi(\beta,\lambda,h) \equiv \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,h)$. The k limit.

c) $m_*(\beta,\lambda) \equiv \frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}$ The h derivative.

a) The $n\to\infty$ limit

$$S(k,\beta,\lambda,h) = \operatorname*{ext}_{\vec{\mu},\ Q_{ii}=1,\ Q\succeq 0} U(\mu, Q)$$

$$\left(\begin{array}{l}\theta \sim \mathrm{Unif}(\{\pm1\}) \\ \sigma \sim \mathrm{Unif}(\{\pm1\}^k)\end{array}\right)$$

$$U(\vec{\mu},Q) = -\beta\lambda\sum_{a=1}^k \mu_a^2 - \beta^2\sum_{ab=1}^k q_{ab}^2$$
$$+ \log \mathbb{E}_{\theta,\sigma}\Big[\exp\Big\{2\beta\lambda\sum_{a=1}^k \mu_a\sigma_a\theta + 2\beta^2\sum_{ab=1}^k q_{ab}\sigma_a\sigma_b + \beta h\sum_{a=1}^k \psi(\sigma_a,\theta)\Big\}\Big].$$

b) The $k\to 0$ limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0}\frac{1}{k}\operatorname*{ext}_{Q\succeq 0,\ Q_{ii}=1,\ \vec{\mu}} U(\vec{\mu},Q).$$

Trick 0: Replica symmetric ansatz.

$$\begin{cases} \mu_a = \mu, & 1\le a\le k, \\ q_{ab} = q, & 1\le a\ne b\le k. \end{cases}$$

$$U(\vec{\mu},Q) = -\beta\lambda k\mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k$$
$$+ \log\mathbb{E}_{\sigma,\theta}\Big[\exp\Big\{\beta\lambda\mu\sum_{a=1}^k\sigma_a\theta + \underline{2\beta^2 q\Big(\sum_{a=1}^k\sigma_a\Big)^2} + \beta h\sum_{a=1}^k\psi(\sigma_a,\theta)\Big\}\Big]$$

Trick 1: $G \sim N(0,1)$, $\mathbb{E}_G\left[e^{\lambda G \sum_{a=1}^{k} \sigma_a}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_{a=1}^{k} \sigma_a\right)^2\right\}$

$$= -\beta\lambda k \mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k$$
$$+ \log \mathbb{E}_{G,\sigma,\theta}\left[\exp\left\{2\beta\lambda\mu \sum_{a=1}^{k} \sigma_a \theta + 2\beta\sqrt{q}\, G \sum_{a=1}^{k} \sigma_a + \beta h \sum_{a=1}^{k} \psi(\sigma_a, \theta)\right\}\right]$$

$$= -\beta\lambda k \mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k \qquad \left(\sigma, \theta \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}\{\pm 1\}\right)$$
$$+ \log \mathbb{E}_{G,\theta}\left[\left(\mathbb{E}_\sigma\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\, G\sigma + \beta h \psi(\sigma,\theta)\right\}\right]\right)^k\right]$$

Trick 2: $\lim_{k\to 0} \frac{1}{k} \log \mathbb{E}_X[Z(X)^k] = \mathbb{E}_X[\log Z(X)]$.

$$\Rightarrow \quad \lim_{k\to 0} \frac{1}{k} U(\mu, q) = -\beta\lambda\mu^2 \underbrace{- \beta^2(1-q^2) + 2\beta^2(1-q)}_{= +\beta^2(1-q)^2}$$
$$+ \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_\sigma\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\, G\sigma + \beta h \psi(\sigma,\theta)\right\}\right]\right\}$$
$$\equiv u(\mu, q; \beta, \lambda, h). \qquad (G, \theta, \sigma) \sim N(0,1) \times \mathrm{Unif}(\{\pm 1\}) \times \mathrm{Unif}(\{\pm 1\})$$

$$\Rightarrow \quad \varphi(\beta, \lambda, h) = \operatorname*{ext}_{\mu, q}\ u(q, \mu; \beta, \lambda, h)$$

$$u(q, \mu; \beta, \lambda, h) = -\beta\lambda\mu^2 + \beta^2(1-q)^2$$
$$+ \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_\sigma\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\, G\sigma + \beta h \psi(\sigma,\theta)\right\}\right]\right\}$$

c) The h derivative

$$\frac{1}{\beta}\partial_h \varphi(\beta, \lambda, h)\Big|_{h=0} = \frac{1}{\beta}\partial_h u(\mu, q; \beta, \lambda, h)\Big|(q = q_*, \mu = \mu_*, h = 0)$$
$$= \mathbb{E}_{G,\theta}\left[\frac{\mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q_*}\, G\sigma\}\psi(\sigma,\theta)\right]}{\mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q_*}\, G\sigma\}\right]}\right]$$
$$= \mathbb{E}_{G,\theta}\left[\mathbb{E}_{\bar{\sigma}\sim D}\left[\psi(\sigma, \theta)\right]\right], \qquad \mathbb{E}[\bar{\sigma}] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\, G)$$

$$(\mu_*, q_*) = \operatorname*{arg\,ext}_{\mu, q}\ u(\mu, q; \beta, \lambda, h)\Big|_{h=0}.$$

$$\Rightarrow \begin{cases} \mu_* = \mathbb{E}_{G,\theta}\left[\theta \cdot \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\, G)\right]. \\ q_* = \mathbb{E}_{G,\theta}\left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\, G)^2\right]. \end{cases}$$

④ The free energy trick and replica trick for $S_*(\beta,\lambda)$.

$$\boxed{\text{Our goal: } S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big].}$$

Observation: Let $(\sigma^1, \sigma^2, \cdots \sigma^N) \sim P_{\beta,\lambda}^{\otimes N}$.

$\Rightarrow$ fix i, $\displaystyle\frac{1}{N}\sum_{a=1}^N \sigma_i^a \xrightarrow{N\to\infty} \langle\sigma_i\rangle_{\beta,\lambda}$ (Law of large numbers).

$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big) \xrightarrow{N\to\infty} \frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)$$

Define $\displaystyle\langle f(\sigma_1, \cdots \sigma_N)\rangle_{\beta,\lambda,N} \equiv \int_{\Omega^{\otimes N}} f(\sigma_1, \cdots, \sigma_N) \prod_{a=1}^N P_{\beta,\lambda}(d\sigma)$.

$$S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big]/n = \lim_{n\to\infty}\lim_{N\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]/n$$

$$\overset{?}{=} \lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]/n$$

Free energy trick:

"E-replica"

$\underline{\Omega} = \Omega^{\otimes N}$, $\quad \underline{\nu_0} = \nu_0^{\otimes N}$, $\quad \underline{\sigma} = (\sigma^1, \sigma^2, \cdots \sigma^N)$.

$$H_{\lambda,h,N}(\underline{\sigma}) = \sum_{a=1}^N H_\lambda(\sigma^a) - h\sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)$$

$$\varphi(\beta,\lambda,h,N) = \lim_{n\to\infty} \frac{1}{n}\log\int_{\underline{\Omega}} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\}\, \underline{\nu_0}(d\underline{\sigma}) \quad ☆$$

$$\partial_h \varphi(\beta,\lambda,0,N) = \lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]/n.$$

We expect:
$$S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big]/n$$
$$= \lim_{N\to\infty} \partial_h\varphi(\beta,\lambda,0,N).$$

In fact, we will show that:

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \frac{1}{n}\sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda}\Big] = \partial_h\varphi(\beta,\lambda,0,N)$$

$$= \mathbb{E}_{G,\theta}\Big[\mathbb{E}_{\bar{\sigma}^a \overset{i.i.d.}{\sim} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))}\Big[\psi\Big(\frac{1}{N}\sum_{a=1}^N \bar{\sigma}^a, \theta\Big)\Big]\Big].$$

$$\Rightarrow \quad \lim_{N\to\infty} \partial_h\varphi(\beta,\lambda,0,N) = \mathbb{E}_{G,\theta}\big[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta)\big].$$

Replica trick.

the n limit.

a). $S(k,\beta,\lambda,h,N) = \lim\limits_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(\int_{\Omega} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\} \underline{\nu_0}(d\underline{\sigma})\right)^k.$

b). $\varphi(\beta,\lambda,h,N) = \lim\limits_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,N).$ the k limit.

c). $S_*(\beta,\lambda) = \lim\limits_{N\to\infty} \partial_h \varphi(\beta,\lambda,0,N).$ the h derivatives.

a) The n limit.

$$H_{\beta,\lambda,N}(\underline{\sigma}) = \sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^a) - h\sum_{i=1}^{n} \psi\left(\sum_{a=1}^{N}\sigma_i^a, \theta_i\right)$$

$\mathbb{E}[Z_n]^k$

$$= \mathbb{E}\left[\int_{\Omega^{\otimes Nk}} \exp\left\{-\beta\sum_{b=1}^{k}\left(\sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^{ab}) + \beta h\sum_{i=1}^{n}\psi\left(\sum_{a=1}^{N}\sigma_i^{ab},\theta_i\right)\right)\right\} \prod_{a=1}^{N}\prod_{b=1}^{k}\nu_0(d\sigma^{ab})\right.$$

Two types of replicas, form k groups.

replicas introduced in the replica trick	σ^{11} - -	σ^{1N}	Group 1
	σ^{21} - - -	σ^{2N}	Group 2
	$\vdots$		$\vdots$
	σ^{k1} - - - - -	σ^{kN}	Group k.

auxilliary "A-replicas" $\longrightarrow$

replicas introduced in the free energy trick "E-replicas"

If $h=0$: all the replicas are symmetric.

If $h\neq 0$: symmetry is broken.

Invariant if exchange replicas in the same group ✓

Invariant if exchange groups ✓

Non-invariant if exchange replicas in different groups. ✗

$$S(k,\beta,\lambda,h,N) = \sup_{\substack{\vec{\mu},\, Q\geq 0\\ Q_{ii}=1}} U(\vec{\mu}, Q)$$

$$U(\vec{\mu},Q) = -\beta\lambda\sum_{b=1}^{N}\sum_{a=1}^{k}\mu_{ab}^2 - \beta^2\sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k} q_{ab,a'b'}^2$$

$$+\log\mathbb{E}_{\sigma\theta}\left[\exp\left\{2\beta\lambda\sum_{b=1}^{k}\sum_{a=1}^{N}\mu_{ab}\sigma^{ab}\theta + 2\beta^2\sum_{bb'=1}^{k}\sum_{aa'=1}^{N} q_{ab,a'b'}\sigma^{ab}\sigma^{a'b'} + \beta h\sum_{b=1}^{k}\psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma_i^{ab},\theta\right)\right\}\right]$$

$$\sigma = (\sigma^{ab})_{\substack{1\le a\le N \\ 1\le b\le k}} \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm 1\}) \quad , \quad \theta \sim \mathrm{Unif}(\{\pm 1\})$$

$$Q = \begin{bmatrix} q_{11,11} & \cdots & q_{11,Nk} \\ \vdots & & \\ q_{Nk,11} & \cdots & q_{Nk,Nk} \end{bmatrix} \qquad \vec{\mu} = (\mu_{11}, \cdots, \mu_{Nk}).$$

b) The $k \to 0$ limit.

Replica symmetric ansatz $\quad (k=3 \ , N=2)$

$$Q = \begin{array}{c} \left[\begin{array}{cc|cc|cc} 1 & q_1 & q_0 & & q_0 & \\ q_1 & 1 & & & & \\ \hline & & 1 & q_1 & & \\ q_0 & & q_1 & 1 & q_0 & \\ \hline q_0 & & q_0 & & 1 & q_1 \\ & & & & q_1 & 1 \end{array}\right] \\ \underbrace{\qquad}_{\text{Group 1}} \underbrace{\qquad}_{\text{Group 2}} \underbrace{\qquad}_{\text{Group 3}} \end{array} \begin{array}{l} \} \text{ Group 1} \\ \} \text{ Group 2} \\ \} \text{ Group 3} \end{array}$$

$$\vec{\mu} = (\mu, \mu, \mu, \cdots \mu)$$

$$U(k, q_1, q_2, \mu, N) = -\beta\lambda Nk - \beta^2 \left[Nk(1-q_1)^2 + N^2 k (q_1 - q_0)^2 + N^2 k^2 q_0^2 - 2(1-q_1)Nk\right]$$

$$T \equiv \left\{ + \log \mathbb{E}_{\sigma,\theta}\left[\exp\left\{ 2\beta\lambda\mu \sum_{ab} \sigma^{ab}\theta + 2\beta^2 (q_1 - q_0) \sum_{b=1}^{k} \left(\sum_{a=1}^{N} \sigma^{ab}\right)^2 + \boxed{2\beta^2 q_0 \left(\sum_{b=1}^{k}\sum_{a=1}^{N} \sigma^{ab}\right)^2} + \beta h \sum_{b=1}^{k} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\right) \right\} \right] \right.$$

$$T = \log \mathbb{E}_{\sigma, \boxed{G_0}, \theta}\left[\exp\left\{ 2\beta\lambda\mu \sum_{ab} \sigma^{ab}\theta + 2\beta^2 (q_1 - q_0) \sum_{b=1}^{k} \left(\sum_{a=1}^{N} \sigma^{ab}\right)^2 + \boxed{2\beta\sqrt{q_0}\, G_0 \sum_{b=1}^{k}\sum_{a=1}^{N} \sigma^{ab}} + \beta h \sum_{b=1}^{k} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\right) \right\} \right]$$

$\sigma \in \{\pm 1\}^{nk}$

$$= \log \mathbb{E}_{\theta, G_0}\left[\left(\mathbb{E}_{\sigma} \exp\left\{ 2\beta\lambda\mu \sum_{a=1}^{N} \sigma^a \theta + \boxed{2\beta^2 (q_1 - q_0) \left(\sum_{a=1}^{N} \sigma^a\right)^2} + 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{N} \sigma^a + \beta h\, \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right) \right\} \right)^k \right]$$

$\sigma \in \{\pm 1\}^N$

$$= \log \mathbb{E}_{\theta,G_0}\Big[\Big(\mathbb{E}_{\sigma,\boxed{G_1}} \exp\Big\{2\beta\lambda\mu \sum_{a=1}^{N} \sigma^a \theta + \boxed{2\beta\sqrt{q_1-q_0}\, G_1 \sum_{a=1}^{N}\sigma^a}$$
$$+ 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{N} \sigma^a + \beta h\, \psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\Big)\Big\}\Big)^k\Big]$$

$$\varphi(\beta,\lambda,h,N) = \lim_{k\to 0} \sup_{q_1,q_0,\mu} \frac{1}{k} U(k,q_1,q_0,\mu,N) = \operatorname{ext}_{\mu,q_1,q_0} u(q_1,q_0,\mu;\beta,\lambda,h,N)$$

$$u(q_1,q_0,\mu;\beta,\lambda,h,N) \equiv \lim_{k\to 0} \frac{1}{k} U(k,q_1,q_0,\mu,N)$$

$$= -\beta\lambda N - \beta^2\left[N(1-q_1)^2 + N^2(q_1-q_0)^2 - 2(1-q_1)N\right]$$
$$+ \mathbb{E}_{\theta,G_0}\Big[\log \mathbb{E}_{\sigma,G_1}\Big[\exp\Big\{2\beta\lambda\mu\sum_{a=1}^{N}\sigma^a\theta + 2\beta\sqrt{q_1-q_0}\, G_1 \sum_{a=1}^{N}\sigma^a$$
$$+ 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{N}\sigma^a + \beta h\, \psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\Big)\Big]\Big].$$

c). $\partial_h \varphi(\beta,\lambda,h,N) = \partial_h u(q_1,q_0,\mu;\beta,\lambda,0,N)$

$$= \mathbb{E}_{\theta,G_0}\left[\frac{\mathbb{E}_{\sigma,G_1}\Big[\exp\Big\{2\beta\lambda\mu_*\sum_{a=1}^{N}\sigma^a\theta + 2\beta\big(\sqrt{q_{1*}-q_{0*}}\,G_1 + \sqrt{q_{0*}}\,G_0\big)\sum_{a=1}^{N}\sigma^a\Big\}\, \psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^a,\theta\Big)\Big]}{\mathbb{E}_{\sigma,G_1}\Big[\exp\Big\{2\beta\lambda\mu_*\sum_{a=1}^{N}\sigma^a\theta + 2\beta\big(\sqrt{q_{1*}-q_{0*}}\,G_1 + \sqrt{q_{0*}}\,G_0\big)\sum_{a=1}^{N}\sigma^a\Big\}\Big]}\right].$$

where $\mu_*, q_{1*}, q_{0*} = \arg\operatorname{ext}_{\mu,q_1,q_0} u(q_1,q_0,\mu;\beta,\lambda,0,N)$.

"Obviously", there exists a stationery point s.t.

$$q_{1*} = q_{0*} = q_*.$$

the $N=1$ formula

$$(\mu_*, q_*) = \arg\operatorname{ext}_{q,\mu} u(q,\mu;\beta,\lambda,0)$$

This is the correct stationery point for some region (β,λ).

The replica symmetric phase.

$$= \mathbb{E}_{\theta,G}\Big[\mathbb{E}_{\sigma^a \overset{\text{i.i.d.}}{\sim} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G))}\Big[\psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^a,\theta\Big)\Big]\Big]$$

$$\Rightarrow S_*(\beta,\lambda) = \lim_{N\to\infty} \partial_h \varphi(\beta,\lambda,0,N) = \mathbb{E}_{\theta,G}\left[\psi\left(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G),\theta\right)\right].$$

④ 1-RSB prediction of $\mathbb{Z}_2$ sync free entropy density.

$$\Omega = \mathbb{Z}_2^n, \qquad \nu_0 = \text{Unif}$$

$$H_{\lambda,h}(\sigma) = -\langle \sigma, W\sigma \rangle - \lambda \langle \sigma, \theta \rangle^2 / n$$

$$Z_n(\beta,\lambda,h) = \int_\Omega \exp\{-\beta H_{\lambda,h}(\sigma)\}\, \nu_0(d\sigma).$$

$$\varphi(\beta,\lambda,h) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log Z_n(\beta,\lambda,h)].$$

a) $S(k,\beta,\lambda,h) \equiv \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[Z_n(\beta,\lambda,h)^k]$. The n limit.

b) $\varphi(\beta,\lambda,h) \equiv \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,h)$. The k limit.

a) The $n\to\infty$ limit

$$S(k,\beta,\lambda,h) = \underset{Q\in\mathbb{R}^{(k+1)\times(k+1)}}{\text{ext}} U(Q).$$

$$U(Q) = -\beta\lambda \sum_{a=1}^k \mu_a^2 - \beta^2 \sum_{ab=1}^k q_{ab}^2$$

$$+ \log \mathbb{E}_{\sigma,\theta}\left[\exp\left\{2\beta\lambda \sum_{a=1}^k \mu_a \sigma_a \theta + 2\beta^2 \sum_{ab=1}^k q_{ab}\sigma_a\sigma_b\right\}\right].$$

b) The $k\to 0$ limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0} \frac{1}{k} \underset{\substack{\mu \\ Q\succeq 0 \\ Q_{ii}=1}}{\text{ext}} U(\vec{\mu}, Q).$$

Replica symmetric ansatz doesn't always hold.

Here we assume 1-step replica symmetric breaking ansatz.

$$Q = \begin{array}{c} \text{size } k_1 \\ \\ \\ \end{array} \left[\begin{array}{cc|cc|cc} 1 & q_1 & q_0 & & q_0 & \\ q_1 & 1 & & & & \\ \hline q_0 & & 1 & q_1 & q_0 & \\ & & q_1 & 1 & & \\ \hline q_0 & & q_0 & & 1 & q_1 \\ & & & & q_1 & 1 \end{array}\right] \begin{array}{l} \} \text{ Group 1} \\ \\ \} \text{ Group 2} \\ \\ \} \text{ Group} \\ \\ \end{array}$$

$$\begin{array}{ccc} \text{Group 1} & \text{Group 2} & \text{Group } k_0 \end{array}$$

$$k = k_1 \times k_0$$

$$\vec{\mu} = (\mu, \mu, \cdots \mu).$$

$$U(\mu, q_0, q_1, k_0, k) = -\beta\lambda k \mu^2 - \beta^2 \left[k(k-k_0) q_0^2 + k(k_0 - 1) q_1^2 + k - 2(1-q_1)k \right]$$

$$T = \left\{ + \log \mathbb{E}_{\sigma,\theta} \left[\exp\left\{ 2\beta\lambda\mu \sum_{b=1}^{k_1} \sum_{a=1}^{k_0} \sigma^{ab}\theta + \boxed{2\beta^2 q_0 \left(\sum_{b=1}^{k_1} \sum_{a=1}^{k_0} \sigma^{ab} \right)^2} + 2\beta^2 (q_1 - q_0) \sum_{b=1}^{k_1} \left(\sum_{a=1}^{k_0} \sigma^{ab} \right)^2 \right] \right.$$

$$T = \log \mathbb{E}_{\sigma,\theta,\boxed{G_0}} \left[\exp\left\{ 2\beta\lambda\mu \sum_{b=1}^{k_1} \sum_{a=1}^{k_0} \sigma^{ab}\theta + \boxed{2\beta\sqrt{q_0}\, G_0 \sum_{b=1}^{k_1} \sum_{a=1}^{k_0} \sigma^{ab}} + 2\beta^2 (q_1 - q_0) \sum_{b=1}^{k_1} \left(\sum_{a=1}^{k_0} \sigma^{ab} \right)^2 \right]$$

$$= \log \mathbb{E}_{\theta,G_0} \left[\left(\mathbb{E}_{\sigma} \exp\left\{ 2\beta\lambda\mu \sum_{a=1}^{k_0} \sigma^a \theta + 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{k_0} \sigma^a + \boxed{2\beta^2 (q_1 - q_0) \left(\sum_{a=1}^{k_0} \sigma^a \right)^2} \right\} \right)^{k_1} \right]$$

$$= \log \mathbb{E}_{\theta,G_0} \left[\left(\mathbb{E}_{\sigma,\boxed{G_1}} \left[\exp\left\{ 2\beta\lambda\mu \sum_{a=1}^{k_0} \sigma^a \theta + 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{k_0} \sigma^a + \boxed{2\beta\sqrt{q_1 - q_0}\, G_1 \sum_{a=1}^{k_0} \sigma^a} \right\} \right] \right)^{k_1} \right]$$

$$= \log \mathbb{E}_{\theta,G_0} \left[\left(\mathbb{E}_{G_1} \left[\mathbb{E}_{\sigma} \left(\exp\left\{ 2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q_0}\, G_0 \sigma + 2\beta\sqrt{q_1 - q_0}\, G_1 \sigma \right\} \right)^{k_0} \right] \right)^{k_1} \right]$$

$$= \log \mathbb{E}_{\theta,G_0} \left[\left(\mathbb{E}_{G_1} \left[\cosh\left(2\beta\lambda\mu\theta + 2\beta\sqrt{q_0}\, G_0 + 2\beta\sqrt{q_1 - q_0}\, G_1 \right)^{k_0} \right] \right)^{k/k_0} \right]$$

$$\Rightarrow \lim_{k\to 0} \frac{1}{k} U(\mu, q_0, q_1, k_0, k) \equiv u(\mu, q_0, q_1, k_0, \beta, \lambda).$$

$$= -\beta\lambda\mu^2 - \beta^2 \left[-k_0 q_0^2 + (k_0 - 1) q_1^2 - 1 - 2(1 - q_1) \right]$$

$$+ \frac{1}{k_0} \mathbb{E}_{\theta,G_0} \left[\log \mathbb{E}_{G_1} \left[\cosh\left(2\beta\lambda\mu\theta + 2\beta\sqrt{q_0}\, G_0 + 2\beta\sqrt{q_1 - q_0}\, G_1 \right)^{k_0} \right] \right]$$

$$\Rightarrow \varphi(\beta, \lambda) = \operatorname*{ext}_{\mu, q_0, q_1, k_0} u(\mu, q_0, q_1, k_0; \beta, \lambda)$$

range of $k_0 \in [0, 1]$.

Remark:

2-RSB ansatz.

$$Q = \left(\begin{array}{cc|cc|c|c}
1 & q_1 & \multicolumn{2}{c|}{\multirow{2}{*}{q_2}} & \multicolumn{2}{c}{\multirow{4}{*}{q_3}} \\
q_1 & 1 & & & & \\
\cline{1-4}
\multicolumn{2}{c|}{\multirow{2}{*}{q_2}} & 1 & q_1 & & \\
& & q_1 & 1 & & \\
\hline
\multicolumn{4}{c|}{\multirow{4}{*}{q_3}} & \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix} & q_2 \\
\cline{5-6}
& & & & q_2 & \begin{matrix} 1 & q_1 \\ q_1 & 1 \end{matrix}
\end{array}\right)$$

The ground state of SK model when $\lambda = 0$ is ∞-RSB.