Mean Field Asymptotics in Statistical Learning.

Mar 1

Lecture 11. $\mathbb{Z}_2$ synchronization and replica symmetry breaking.

① $\mathbb{Z}_2$ synchronization.

Signal: $\theta \in \mathbb{R}^n$, $\theta_i \sim_{i.i.d.} \mathrm{Unif}(\{\pm 1\})$. $\lambda \geq 0$.

Observation: $Y = \frac{\lambda}{n}\theta\theta^T + W \in \mathbb{R}^{n\times n}$. $W \sim GOE(n)$.

Estimator: $\hat{\theta}(Y) = \langle \sigma \rangle_{\beta,\lambda} \equiv \sum_{\sigma \in \mathbb{Z}_2} \sigma P_{\beta,\lambda}(\sigma)$.

$$\begin{cases} \text{MLE} & \beta = \infty \\ \text{Bayes} & \beta = \lambda/2 \end{cases}$$

Gibbs measure: $P_{\beta,\lambda}(\sigma) \propto \exp\{\beta \langle \sigma, Y\sigma\rangle\}\nu_0(\sigma)$

An issue: $P_{\beta,\lambda}(\sigma) = P_{\beta,\lambda}(-\sigma)$, $\Rightarrow$ $\langle\sigma\rangle_{\beta,\lambda} = 0$.

A solution: Prior ν_ε on θ_i

$$\theta_i = \begin{cases} 1, & \text{w.p. } \frac{1+\varepsilon}{2}. \\ -1, & \text{w.p. } \frac{1-\varepsilon}{2}. \end{cases}$$

ε is very small.

$$P_{\beta,\lambda,\varepsilon}(\sigma) \propto \exp\{\beta\langle\sigma, Y\sigma\rangle\}\nu_\varepsilon(\sigma)$$

Limiting observables $\qquad \psi: \mathbb{R}^2 \to \mathbb{R}$, sufficiently smooth

$$m_*(\beta,\lambda) \equiv \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \frac{1}{n}\sum_{i=1}^n \psi(\sigma_i, \theta_i)\Big\rangle_{\beta,\lambda,\varepsilon}\Big].$$

$$s_*(\beta,\lambda) \equiv \lim_{\varepsilon\to 0+}\lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda,\varepsilon}, \theta_i)\Big].$$

Ignore this ε issue for now.

This doesn't affect much the replica calculation.

Formalism (all the results are in the sense of taking $P_{\beta,\lambda,\varepsilon}$ and then send $\varepsilon \to 0$).

(A) $$m_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\left[\left\langle \sum_{i=1}^n \psi(\sigma_i, \theta_i) \right\rangle_{\beta,\lambda}\right] / n$$

$$= \mathbb{E}_{\underset{N(0,1)}{\underset{\uparrow}{G}},\underset{\mathrm{Unif}(\{\pm1\})}{\underset{\uparrow}{\theta}}} \left\{ \mathbb{E}_{\bar{\sigma} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))}\left[\psi(\bar{\sigma}, \theta)\right]\right\} \quad \begin{pmatrix} \bar{\sigma} \in \{\pm 1\} \\ \mathbb{E}[\bar{\sigma}] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G) \end{pmatrix}$$

$$q_* = \mathbb{E}_{G,\theta}\left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)^2\right]$$

$$\mu_* = \mathbb{E}_{G,\theta}\left[\theta \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)\right]. \qquad \mu_* > 0.$$

Actually: $\frac{1}{n}\sum_{i=1}^n \psi(\sigma_i, \theta_i) \longrightarrow \mathbb{E}[\psi(\bar{\sigma}, \theta)]$ in probability under $\mathbb{E}[\langle \cdot \rangle_{\beta,\lambda}]$

Interpretation: $\frac{1}{n}\sum_{i=1}^n \delta_{(\sigma_i,\theta_i)} \longrightarrow$ Law of $(\bar{\sigma}, \theta)$

$\theta \sim \mathrm{Unif}(\{\pm1\})$, $G \sim N(0,1)$ $\boxed{\bar{\sigma}} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))$

(B) $$s_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\left[\psi(\langle \sigma_i \rangle_{\beta,\lambda}, \theta_i)\right] / n$$

Need more work to show this.

$$= \mathbb{E}_{G,\theta}\left[\psi\left(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta\right)\right].$$

Interpretation: $\frac{1}{n}\sum_{i=1}^n \delta_{(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)} \to$ Law of (m, θ)

$\theta \sim \mathrm{Unif}(\{\pm1\})$, $G \sim N(0,1)$, $\boxed{m} = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)$

$m = \mathbb{E}[\bar{\sigma}]$.

A cheating principle:

When replica symmetric ansatz holds, suppose we have

$\forall \psi: \mathbb{R}^2 \to \mathbb{R}$, smooth test function

$$\lim_{N\to\infty} \mathbb{E}\left[\left\langle \frac{1}{n}\sum_{i=1}^n \psi(\sigma_i, \theta_i)\right\rangle_\beta\right] = \mathbb{E}_{\theta,X}\left[\mathbb{E}_{\bar{\sigma}\sim D(\theta,X)}[\psi(\bar{\sigma}, \theta)]\right]$$

$\Rightarrow$ $\forall \psi: \mathbb{R}^2 \to \mathbb{R}$, smooth test function

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_\beta, \theta_i)\right] = \mathbb{E}_{\theta,X}\left[\psi(\mathbb{E}_{\bar{\sigma}\sim D(\theta,X)}[\bar{\sigma}], \theta)\right].$$

② Free energy trick and replica trick for $m_*(\beta,\lambda)$. A quick review of the last lecture.

perturbation.

$$\Omega = \mathbb{Z}_2^n, \quad \nu_0 = \text{Unif}$$

$$H_{\lambda,h}(\sigma) = -\langle \sigma, W\sigma\rangle - \lambda \langle \sigma, \theta\rangle^2/n - h\sum_{i=1}^n \psi(\sigma_i,\theta_i).$$

$$Z_n(\beta,\lambda,h) = \int_\Omega \exp\{-\beta H_{\lambda,h}(\sigma)\}\,\nu_0(d\sigma).$$

$$\varphi(\beta,\lambda,h) = \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\log Z_n(\beta,\lambda,h)].$$

$$m_*(\beta,\lambda) = \frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}.$$

a) $S(k,\beta,\lambda,h) \equiv \lim_{n\to\infty}\frac{1}{n}\log \mathbb{E}[Z_n(\beta,\lambda,h)^k]$. The n limit.

b) $\varphi(\beta,\lambda,h) \equiv \lim_{k\to 0}\frac{1}{k} S(k,\beta,\lambda,h)$. The k limit.

c) $m_*(\beta,\lambda) \equiv \frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}$ The h derivative.

a) The $n\to\infty$ limit

$$S(k,\beta,\lambda,h) = \underset{\substack{\vec{\mu}\in\mathbb{R}^k,\ Q_{ii}=1\\ Q\succeq 0}}{\text{ext}}\ U(\vec{\mu},Q) \qquad Q\in\mathbb{R}^{k\times k}$$

$$\begin{pmatrix}\theta\sim\text{Unif}(\{\pm1\})\\ \sigma\sim\text{Unif}(\{\pm1\}^k)\end{pmatrix}$$

$$U(\vec{\mu},Q) = -\beta\lambda\sum_{a=1}^k \mu_a^2 - \beta^2\sum_{ab=1}^k q_{ab}^2$$
$$+\log \mathbb{E}_{\theta,\sigma}\Big[\exp\Big\{2\beta\lambda\sum_{a=1}^k \mu_a\sigma_a\theta + 2\beta^2\sum_{ab=1}^k q_{ab}\sigma_a\sigma_b + \beta h\sum_{a=1}^k \psi(\sigma_a,\theta)\Big\}\Big].$$

b) The $k\to 0$ limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0}\frac{1}{k}\ \underset{\substack{Q\succeq 0\\ Q_{ii}=1\\ \vec{\mu}}}{\text{ext}}\ U(\vec{\mu},Q).$$

Trick 0: Replica symmetric ansatz.

$$\begin{cases}\mu_a = \mu, & 1\le a\le k,\\ q_{ab} = q, & 1\le a\ne b\le k.\end{cases}$$

$$U(\vec{\mu},Q) = -\beta\lambda k\mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k$$
$$+\log\mathbb{E}_{\sigma,\theta}\Big[\exp\Big\{\beta\lambda\mu\sum_{a=1}^k\sigma_a\theta + \underline{2\beta^2 q\Big(\sum_{a=1}^k\sigma_a\Big)^2} + \beta h\sum_{a=1}^k\psi(\sigma_a,\theta)\Big\}\Big]$$

Trick 1: $G\sim N(0,1)$, $\mathbb{E}_G\left[e^{\lambda G\sum_{a=1}^{k}\sigma_a}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_{a=1}^{k}\sigma_a\right)^2\right\}$

$$= -\beta\lambda k\mu^2 - \beta^2\left(k + k(k-1)q^2\right) + 2\beta^2(1-q)k$$
$$+ \log \mathbb{E}_{G,\sigma,\theta}\left[\exp\left\{2\beta\lambda\mu\sum_{a=1}^{k}\sigma_a\theta + 2\beta\sqrt{q}\,G\sum_{a=1}^{k}\sigma_a + \beta h\sum_{a=1}^{k}\psi(\sigma_a,\theta)\right\}\right]$$

$$= -\beta\lambda k\mu^2 - \beta^2\left(k + k(k-1)q^2\right) + 2\beta^2(1-q)k \qquad \left(\sigma,\theta \overset{iid}{\sim} \mathrm{Unif}\{\pm 1\}\right)$$
$$+ \log \mathbb{E}_{G,\theta}\left[\left(\mathbb{E}_{\sigma}\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\,G\sigma + \beta h\psi(\sigma,\theta)\right\}\right]\right)^k\right]$$

Trick 2: $\lim_{k\to 0}\frac{1}{k}\log \mathbb{E}_X\left[Z(X)^k\right] = \mathbb{E}_X\left[\log Z(X)\right]$.

$$\Rightarrow \quad \lim_{k\to 0}\frac{1}{k}U(\mu,q) = -\beta\lambda\mu^2 \underbrace{-\beta^2(1-q^2) + 2\beta^2(1-q)}_{=+\beta^2(1-q)^2}$$
$$+ \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_{\sigma}\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\,G\sigma + \beta h\psi(\sigma,\theta)\right\}\right]\right\}$$
$$\equiv u(\mu,q;\beta,\lambda,h). \qquad (G,\theta,\sigma)\sim N(0,1)\times \mathrm{Unif}(\{\pm 1\})\times \mathrm{Unif}(\{\pm 1\})$$

$$\Rightarrow \quad \varphi(\beta,\lambda,h) = \operatorname*{ext}_{\mu,q}\ u(q,\mu;\beta,\lambda,h)$$

$$u(q,\mu;\beta,\lambda,h) = -\beta\lambda\mu^2 + \beta^2(1-q)^2$$
$$+ \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_{\sigma}\left[\exp\left\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\,G\sigma + \beta h\psi(\sigma,\theta)\right\}\right]\right\}$$

c) The h derivative

$$\frac{1}{\beta}\partial_h\varphi(\beta,\lambda,h)\Big|_{h=0} = \frac{1}{\beta}\partial_h u(\mu,q;\beta,\lambda,h)\Big|\left(q=q_*,\ \mu=\mu_*,\ h=0\right)$$
$$= \mathbb{E}_{G,\theta}\left[\frac{\mathbb{E}_{\sigma}\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q}\,G\sigma\}\psi(\sigma,\theta)\right]}{\mathbb{E}_{\sigma}\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q_*}\,G\sigma\}\right]}\right]$$
$$= \mathbb{E}_{G,\theta}\left[\mathbb{E}_{\bar\sigma\sim D}\left[\psi(\bar\sigma,\theta)\right]\right], \qquad \mathbb{E}[\bar\sigma] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G)$$

$$(\mu_*, q_*) = \arg\operatorname*{ext}_{\mu,q}\ u(\mu,q;\beta,\lambda,h)\Big|_{h=0}.$$

$$\Rightarrow \begin{cases} \mu_* = \mathbb{E}_{G,\theta}\left[\theta\cdot\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G)\right]. & \mu_*>0,\ \mu_*<0 \\ q_* = \mathbb{E}_{G,\theta}\left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G)^2\right]. & q_* \end{cases}$$

④ The free energy trick and replica trick for $S_*(\beta,\lambda)$.

$$\boxed{\text{Our goal: } S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big].}$$

Observation: Let $(\sigma^1, \sigma^2, \cdots \sigma^N) \sim P_{\beta,\lambda}^{\otimes N}$. $\quad$ N # of replicas

$\Rightarrow$ fixed i: $\quad \frac{1}{N}\sum_{a=1}^N \sigma_i^a \xrightarrow{N\to\infty} \langle\sigma_i\rangle_{\beta,\lambda}$. (Law of large number).

$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^n \psi(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i) \xrightarrow{N\to\infty} \frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)$

Define $\quad \langle f(\sigma^1, \cdots, \sigma^N)\rangle_{\beta,\lambda,N} = \int_{\Omega^{\otimes N}} f(\sigma^1, \cdots, \sigma^N) \prod_{a=1}^N P_{\beta,\lambda}(d\sigma)$.

$$S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{n}\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big] = \lim_{n\to\infty}\lim_{N\to\infty} \mathbb{E}\Big[\Big\langle\frac{1}{n}\sum_{i=1}^n \psi(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i)\Big\rangle_{\beta,\lambda,N}\Big]$$

$$\overset{?}{=} \lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}[\cdots].$$

Free energy trick:

"E-replica"

$$\underline{\Omega} = \Omega^{\otimes N}. \quad \underline{\nu_0} = \nu_0^{\otimes N}, \quad \underline{\sigma} = (\sigma^1, \sigma^2, \cdots \sigma^N).$$

$$H_{\lambda,h,N}(\underline{\sigma}) = \sum_{a=1}^N H_\lambda(\sigma^a) - h\sum_{i=1}^n \psi(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i)$$

$$\varphi(\beta,\lambda,h,N) = \lim_{n\to\infty} \frac{1}{n}\log\int_{\underline{\Omega}} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\}\,\underline{\nu_0}(d\sigma') \quad ☆$$

$$\partial_h \varphi(\beta,\lambda,0,N) = \lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i)\Big\rangle_{\beta,\lambda,N}\Big]/n.$$

We expect: $\quad S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)\Big]/n$

$$= \lim_{N\to\infty} \partial_h \varphi(\beta,\lambda,0,N).$$

In fact, we will show that:

$$\lim_{n\to\infty} \mathbb{E}\Big[\Big\langle\frac{1}{n}\sum_{i=1}^n \psi(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i)\Big\rangle_{\beta,\lambda}\Big] = \partial_h\varphi(\beta,\lambda,0,N)$$

$$= \mathbb{E}_{G,\theta}\Big[\mathbb{E}_{\bar\sigma^a \overset{i.i.d.}{\sim} D(\tanh(2\beta\lambda\mu_* \theta + 2\beta\sqrt{q_*}G))}\Big[\psi(\frac{1}{N}\sum_{a=1}^N \bar\sigma^a, \theta)\Big]\Big].$$

$$\Rightarrow \quad \lim_{N\to\infty} \partial_h\varphi(\beta,\lambda,0,N) = \mathbb{E}_{G,\theta}[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta)].$$

Replica trick.

the n limit.

a). $S(k,\beta,\lambda,h,N) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(\int_{\Omega} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\} \underline{\nu_0}(d\underline{\sigma})\right)^k$.

the k limit.

b). $\varphi(\beta,\lambda,h,N) = \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,N)$.

the h derivatives.

c). $S_*(\beta,\lambda) = \lim_{N\to\infty} \frac{1}{\beta}\partial_h \varphi(\beta,\lambda,0,N)$.

a) The n limit.

$$H_{\beta,\lambda,N}(\underline{\sigma}) = \sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^a) - h\sum_{i=1}^{n} 4\left(\sum_{a=1}^{N}\sigma_i^a, \theta_i\right)$$

$\mathbb{E}[Z_n^k]$

$$= \mathbb{E}\left[\int_{\Omega^{\otimes Nk}} \exp\left\{-\beta\sum_{b=1}^{k}\left(\sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^{ab}) + \beta h\sum_{i=1}^{n} 4\left(\sum_{a=1}^{N}\sigma_i^{ab},\theta_i\right)\right)\right\}\prod_{a=1}^{N}\prod_{b=1}^{k}\nu_0(d\sigma^{ab})\right]$$

Two types of replicas, form k groups.

replicas introduced in the replica trick	σ^{11} - -	σ^{1N}	Group 1
	σ^{21} - - -	σ^{2N}	Group 2
	$\vdots$		$\vdots$
	σ^{k1} - - - - -	σ^{kN}	Group k.

auxilliary "A-replicas" $\longrightarrow$

replicas introduced in the free energy trick "E-replicas"

If $h=0$: all the replicas are symmetric.

If $h\neq 0$: symmetry is broken.

Invariant if exchange replicas in the same group ✓

Invariant if exchange groups ✓

Non-invariant if exchange replicas in different groups. ✗

$$S(k,\beta,\lambda,h,N) = \underset{\substack{\vec{\mu},\ Q\geq 0\\ Q_{ii}=1}}{\mathrm{ext}}\ U(\vec{\mu}, Q)$$

$$U(\vec{\mu},Q) = -\beta\lambda\sum_{b=1}^{N}\sum_{a=1}^{k}\mu_{ab}^2 - \beta^2\sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k} q_{ab,a'b'}^2$$

$$+\log\mathbb{E}_{\sigma\theta}\left[\exp\left\{2\beta\lambda\sum_{b=1}^{k}\sum_{a=1}^{N}\mu_{ab}\sigma^{ab}\theta + 2\beta^2\sum_{bb'=1}^{k}\sum_{aa'=1}^{N} q_{ab,a'b'}\sigma^{ab}\sigma^{a'b'} + \beta h\sum_{b=1}^{k} 4\left(\frac{1}{N}\sum_{a=1}^{N}\sigma_i^{ab},\theta\right)\right\}\right]$$

$$\sigma = (\sigma^{ab})_{\substack{1\le a\le N\\ 1\le b\le k}} \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm1\}), \quad \theta \sim \mathrm{Unif}(\{\pm1\})$$

$$Q = \begin{bmatrix} q_{11,11} & \cdots & \cdots & q_{11,Nk} \\ \vdots & & & \\ \vdots & & & \\ q_{Nk,11} & \cdots & \cdots & q_{Nk,Nk} \end{bmatrix} \qquad \vec{\mu} = (\mu_{11}, \cdots, \mu_{Nk}).$$

b) The $k \to 0$ limit.

Replica symmetric ansatz $(k=3, N=2)$

$$Q = \left[\begin{array}{cc|cc|cc} 1 & q_1 & q_0 & q_0 & & \\ q_1 & 1 & q_0 & q_0 & \multicolumn{2}{c}{q_0} \\ \hline & & 1 & q_1 & & \\ \multicolumn{2}{c|}{q_0} & q_1 & 1 & \multicolumn{2}{c}{q_0} \\ \hline & & & & 1 & q_1 \\ \multicolumn{2}{c|}{q_0} & \multicolumn{2}{c|}{q_0} & q_1 & 1 \end{array}\right] \begin{array}{l} \left.\right\} \text{Group 1} \\ \left.\right\} \text{Group 2} \\ \left.\right\} \text{Group 3} \end{array}$$

$$\qquad\quad \underbrace{\qquad}_{\text{Group 1}} \underbrace{\qquad}_{\text{Group 2}} \underbrace{\qquad}_{\text{Group 3}}$$

$$\vec{\mu} = (\mu, \mu, \mu, \cdots \mu)$$

$$U(k, q_1, q_2, \mu, N) = -\beta\lambda Nk\mu^2 - \beta^2\left[Nk(1-q_1)^2 + N^2k(q_1-q_0)^2 + N^2k^2q_0^2 - 2(1-q_1)Nk\right]$$

$$T \equiv \left\{ +\log \mathbb{E}_{\sigma,\theta}\left[\exp\left\{2\beta\lambda\mu\sum_{ab}\sigma^{ab}\theta + 2\beta^2(q_1-q_0)\sum_{b=1}^{k}\left(\sum_{a=1}^{N}\sigma^{ab}\right)^2 + \boxed{2\beta^2 q_0\left(\sum_{b=1}^{k}\sum_{a=1}^{N}\sigma^{ab}\right)^2} + \beta h\sum_{b=1}^{k}\Psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\right)\right\}\right]\right.$$

$$\mathbb{E}\left[\exp\left\{\lambda G_0 \sum_b\sum_a \sigma^{ab}\right\}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_b\sum_a\sigma^{ab}\right)^2\right\}$$

$$T = \log \mathbb{E}_{\underset{\uparrow}{\sigma},\boxed{G_0},\theta}\left[\exp\left\{2\beta\lambda\mu\sum_{ab}\sigma^{ab}\theta + 2\beta^2(q_1-q_0)\sum_{b=1}^{k}\left(\sum_{a=1}^{N}\sigma^{ab}\right)^2 + \boxed{2\beta\sqrt{q_0}\, G_0 \sum_{b=1}^{k}\sum_{a=1}^{N}\sigma^{ab}} + \beta h\sum_{b=1}^{k}\Psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\right)\right\}\right]$$

$\sigma \in \{\pm1\}^{Nk}$

$$= \log \mathbb{E}_{\theta,G_0}\left[\left(\mathbb{E}_{\underset{\uparrow}{\sigma}} \exp\left\{2\beta\lambda\mu\sum_{a=1}^{N}\sigma^a\theta + \boxed{2\beta^2(q_1-q_0)\left(\sum_{a=1}^{N}\sigma^a\right)^2} + 2\beta\sqrt{q_0}\, G_0\sum_{a=1}^{N}\sigma^a + \beta h\,\Psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right)\right\}\right)^k\right]$$

$\sigma \in \{\pm1\}^N$

$$= \log \mathbb{E}_{\theta, G_0}\left[\left(\mathbb{E}_{\sigma, \boxed{G_1}} \exp\left\{2\beta\lambda\mu \sum_{a=1}^{N} \sigma^a \theta + \boxed{2\beta\sqrt{q_1 - q_0}\, G_1 \sum_{a=1}^{N} \sigma^a} + 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{N} \sigma^a + \beta h\, \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right)\right\}\right)^k\right]$$

$$\varphi(\beta, \lambda, h, N) = \lim_{k \to 0} \sup_{q_1, q_0, \mu} \frac{1}{k} U(k, q_1, q_0, \mu, N) = \operatorname*{ext}_{\mu, q_1, q_0} u(q_1, q_0, \mu; \beta, \lambda, h, N)$$

$$u(q_1, q_0, \mu; \beta, \lambda, h, N) \equiv \lim_{k \to 0} \frac{1}{k} U(k, q_1, q_0, \mu, N)$$

$$= -\beta\lambda N - \beta^2\left[N(1-q_1)^2 + N^2(q_1 - q_0)^2 - 2(1-q_1)N\right]$$

$$+ \mathbb{E}_{\theta, G_0}\left[\log \mathbb{E}_{\sigma, G_1}\left[\exp\left\{2\beta\lambda\mu \sum_{a=1}^{N}\sigma^a\theta + 2\beta\sqrt{q_1 - q_0}\, G_1 \sum_{a=1}^{N}\sigma^a + 2\beta\sqrt{q_0}\, G_0 \sum_{a=1}^{N}\sigma^a + \beta h\, \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right)\right]\right]\right].$$

c). $\partial_h \varphi(\beta, \lambda, h, N) = \partial_h u(q_1, q_0, \mu; \beta, \lambda, 0, N)$

$$= \mathbb{E}_{\theta, G_0}\left[\frac{\mathbb{E}_{\sigma, G_1}\left[\exp\left\{2\beta\lambda\mu_* \sum_{a=1}^{N}\sigma^a\theta + 2\beta\left(\sqrt{q_{1*} - q_{0*}}\, G_1 + \sqrt{q_{0*}}\, G_0\right)\sum_{a=1}^{N}\sigma^a\right\} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right)\right]}{\mathbb{E}_{\sigma, G_1}\left[\exp\left\{2\beta\lambda\mu_* \sum_{a=1}^{N}\sigma^a\theta + 2\beta\left(\sqrt{q_{1*} - q_{0*}}\, G_1 + \sqrt{q_{0*}}\, G_0\right)\sum_{a=1}^{N}\sigma^a\right\}\right]}\right].$$

where $\mu_*, q_{1*}, q_{0*} = \operatorname*{arg\,ext}_{\mu, q_1, q_0} u(q_1, q_0, \mu; \beta, \lambda, 0, N)$.

"Obviously", there exists a stationery point s.t.

$$q_{1*} = q_{0*} = q_*.$$

$$(\mu_*, q_*) = \operatorname*{arg\,ext}_{q, \mu} u(q, \mu; \beta, \lambda, 0)$$

the $N=1$ formula.

This is the correct stationery point for some region (β, λ).

The replica symmetric phase.

$$= \mathbb{E}_{\theta, G}\left[\mathbb{E}_{\sigma^a \overset{\text{i.i.d.}}{\sim} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\, G))}\left[\psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\right)\right]\right]$$

$$\Rightarrow S_*(\beta, \lambda) = \lim_{N \to \infty} \partial_h \varphi(\beta, \lambda, 0, N) = \mathbb{E}_{\theta, G}\left[\psi\left(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\, G), \theta\right)\right].$$

④ 1-RSB prediction of $\mathbb{Z}_2$ sync free entropy density.

$$\Omega = \mathbb{Z}_2^n, \quad \nu_0 = \mathrm{Unif}$$

$$H_\lambda(\sigma) = -\langle \sigma, W\sigma \rangle - \lambda \langle \sigma, \theta \rangle^2 / n$$

$$Z_n(\beta,\lambda) = \int_\Omega \exp\{-\beta H_\lambda(\sigma)\} \nu_0(d\sigma).$$

$$\varphi(\beta,\lambda) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log Z_n(\beta,\lambda)].$$

a) $S(k,\beta,\lambda,h) \equiv \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[Z_n(\beta,\lambda,h)^k]$. The n limit.

b) $\varphi(\beta,\lambda,h) \equiv \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,h)$. The k limit.

a) The $n \to \infty$ limit

$$S(k,\beta,\lambda,h) = \operatorname{ext} \ U(\vec{\mu}, Q).$$

$$U(\vec{\mu}, Q) = -\beta\lambda \sum_{a=1}^k \mu_a^2 - \beta^2 \sum_{ab=1}^k q_{ab}^2$$
$$+ \log \mathbb{E}_{\sigma,\theta}\left[\exp\left\{2\beta\lambda \sum_{a=1}^k \mu_a \sigma_a \theta + 2\beta^2 \sum_{ab=1}^k q_{ab}\sigma_a\sigma_b\right\}\right].$$

b) The $k \to 0$ limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0} \frac{1}{k} \operatorname*{ext}_{\substack{\mu \\ Q\succeq 0 \\ Q_{ii}=1}} U(\vec{\mu}, Q).$$

Replica symmetric ansatz doesn't always hold.

Here we assume 1-step replica symmetric breaking ansatz.

$$Q = \begin{array}{c} \\ \text{size } k_1 \left\{ \vphantom{\begin{matrix} 1 \\ q_1 \end{matrix}} \right. \\ \\ \\ \\ \\ \end{array} \underbrace{\left[\begin{array}{cc|cc|cc} 1 & q_1 & q_0 & & q_0 & \\ q_1 & 1 & & & & \\ \hline & & 1 & q_1 & & \\ q_0 & & q_1 & 1 & q_0 & \\ \hline q_0 & & q_0 & & 1 & q_1 \\ & & & & q_1 & 1 \end{array}\right]}_{\text{Group 1} \quad \text{Group 2} \quad \text{Group } k_0} \begin{array}{l} \} \text{ Group 1} \\ \\ \} \text{ Group 2} \\ \\ \} \text{ Group} \\ \\ \end{array}$$

$$k = k_1 \times k_0$$

$$\vec{\mu} = (\mu, \mu, \cdots \mu).$$

$$U(\mu, q_0, q_1, k_0, k) = -\beta\lambda k\mu^2 - \beta^2\left[k(k-k_0)q_0^2 + k(k_0-1)q_1^2 + k - 2(1-q_1)k\right]$$

$$T = \begin{cases} +\log \mathbb{E}_{\sigma,\theta}\Big[\exp\Big\{2\beta\lambda\mu \sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\sigma^{ab}\theta \\ \quad + \boxed{2\beta^2 q_0 \Big(\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\sigma^{ab}\Big)^2} + 2\beta^2(q_1-q_0)\sum_{b=1}^{k_1}\Big(\sum_{a=1}^{k_0}\sigma^{ab}\Big)^2\Big] \end{cases}$$

$$T = \log \mathbb{E}_{\sigma,\theta,\boxed{G_0}}\Big[\exp\Big\{2\beta\lambda\mu\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\sigma^{ab}\theta + \boxed{2\beta\sqrt{q_0}\,G_0\sum_{b=1}^{k_1}\sum_{a=1}^{k_0}\sigma^{ab}} + 2\beta^2(q_1-q_0)\sum_{b=1}^{k_1}\Big(\sum_{a=1}^{k_0}\sigma^{ab}\Big)^2\Big]$$

$$= \log \mathbb{E}_{\theta,G_0}\Big[\Big(\mathbb{E}_\sigma \exp\Big\{2\beta\lambda\mu\sum_{a=1}^{k_0}\sigma^a\theta + 2\beta\sqrt{q_0}\,G_0\sum_{a=1}^{k_0}\sigma^a + \boxed{2\beta^2(q_1-q_0)\Big(\sum_{a=1}^{k_0}\sigma^a\Big)^2}\Big\}\Big)^{k_1}\Big]$$

$$= \log \mathbb{E}_{\theta,G_0}\Big[\Big(\mathbb{E}_{\sigma,\boxed{G_1}}\Big[\exp\Big\{2\beta\lambda\mu\sum_{a=1}^{k_0}\sigma^a\theta + 2\beta\sqrt{q_0}\,G_0\sum_{a=1}^{k_0}\sigma^a + \boxed{2\beta\sqrt{q_1-q_0}\,G_1\sum_{a=1}^{k_0}\sigma^a}\Big\}\Big]\Big)^{k_1}\Big]$$

$$= \log \mathbb{E}_{\theta,G_0}\Big[\Big(\mathbb{E}_{G_1}\Big[\mathbb{E}_\sigma\Big(\exp\Big\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q_0}\,G_0\sigma + 2\beta\sqrt{q_1-q_0}\,G_1\sigma\Big\}\Big)^{k_0}\Big]\Big)^{k_1}\Big]$$

$$= \log \mathbb{E}_{\theta,G_0}\Big[\Big(\mathbb{E}_{G_1}\Big[\cosh\big(2\beta\lambda\mu\theta + 2\beta\sqrt{q_0}\,G_0 + 2\beta\sqrt{q_1-q_0}\,G_1\big)^{k_0}\Big]\Big)^{k/k_0}\Big]$$

$$\Rightarrow \lim_{k\to 0}\frac{1}{k}U(\mu, q_0, q_1, k_0, k) \equiv u(\mu, q_0, q_1, k_0, \beta, \lambda).$$

$$= -\beta\lambda\mu^2 - \beta^2\left[-k_0 q_0^2 + (k_0-1)q_1^2 - 1 - 2(1-q_1)\right]$$

$$+ \frac{1}{k_0}\mathbb{E}_{\theta,G_0}\Big[\log \mathbb{E}_{G_1}\Big[\cosh\big(2\beta\lambda\mu\theta + 2\beta\sqrt{q_0}\,G_0 + 2\beta\sqrt{q_1-q_0}\,G_1\big)^{k_0}\Big]\Big]$$

$$\Rightarrow \varphi(\beta,\lambda) = \operatorname*{ext}_{\mu, q_0, q_1, k_0} u(\mu, q_0, q_1, k_0; \beta, \lambda)$$

range of $k_0 \in [0,1]$.

Remark:

$1 \le k_p \le k_{p-1} \le \cdots \le k_0 \le k.$

$0 \le k_0 \le k_1 \le \cdots \quad k_p \le 1.$

2 - RSB ansatz.

$$Q = \left[\begin{array}{cc|cc|cc|cc} 1 & q_1 & \multicolumn{2}{c|}{q_2} & & & & \\ q_1 & 1 & & & \multicolumn{4}{c}{q_3} \\ \hline \multicolumn{2}{c|}{q_2} & 1 & q_1 & & & & \\ & & q_1 & 1 & & & & \\ \hline & & & & 1 & q_1 & \multicolumn{2}{c}{q_2} \\ \multicolumn{4}{c|}{q_3} & q_1 & 1 & & \\ \hline & & & & \multicolumn{2}{c|}{q_2} & 1 & q_1 \\ & & & & & & q_1 & 1 \end{array}\right]$$

The ground state of SK model when $\lambda = 0$ is ∞ - RSB.

$$\mu(dq) = \frac{1}{k(k-1)} \sum_{\substack{a,b=1 \\ a \neq b}}^{k} \delta_{\langle \sigma^a, \sigma^b \rangle / n}.$$

$\lim_{n\to\infty} \mathbb{E} \langle \mu(\quad \cdot \quad) \rangle_{\beta,\lambda,k} \leftarrow \in \mathcal{P}([-1,1]).$

$\rightarrow \mathcal{P}([0,1]).$

$\downarrow$

$\sum_{j=1}^{p} (k_j - k_{j-1}) \delta_{q_j}$