Mean Field Asymptotics in Statistical Learning. Feb 24th.

Lecture 9 Replica methods and $\mathbb{Z}_2$ synchronization.

① $\mathbb{Z}_2$ synchronization.

Signal: $\theta \in \mathbb{R}^n$, $\theta_i \sim_{i.i.d.} \mathrm{Unif}(\{\pm 1\})$, $\lambda \geq 0$.

Observation: $Y = \frac{\lambda}{n}\theta\theta^T + W \in \mathbb{R}^{n\times n}$. $W \sim GOE(n)$.

Estimator: $\hat{\theta}(Y) = \sum_{\sigma \in \mathbb{Z}_2^n} \sigma\, P_{\beta,\lambda}(\sigma)$

Gibbs measure: $P_{\beta,\lambda}(\sigma) \propto \exp\{\beta \langle \sigma, Y\sigma\rangle\}$.

$\begin{cases} \text{MLE} & \beta = \infty \\ \text{Bayes} & \beta = \lambda/2 \end{cases}$

Limiting observables:

$$m_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}[\langle \sum_{i=1}^n \psi(\sigma_i, \theta_i)\rangle_{\beta,\lambda}]/n.$$

$$s_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}[\sum_{i=1}^n \psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)]/n.$$

Formalism (Our goal today)

(A) $$m_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}[\langle \sum_{i=1}^n \psi(\sigma_i, \theta_i)\rangle_{\beta,\lambda}]/n$$

$$= \mathbb{E}_{G,\theta}\{\mathbb{E}_{\bar\sigma\sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))}[\psi(\bar\sigma,\theta)]\} \quad \left(\begin{array}{c}\bar\sigma \in \{\pm 1\} \\ \mathbb{E}[\bar\sigma] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)\end{array}\right)$$

$$q_* = \mathbb{E}_{G,\theta}[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)^2]$$

$$\mu_* = \mathbb{E}_{G,\theta}[\theta \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)].$$

Actually: $\frac{1}{n}\sum_{i=1}^n \psi(\sigma_i,\theta_i) \longrightarrow \mathbb{E}[\psi(\bar\sigma,\theta)]$ in probability under $\mathbb{E}[\langle\cdot\rangle_{\beta,\lambda}]$

Interpretation: $\frac{1}{n}\sum_{i=1}^n \delta_{(\sigma_i,\theta_i)} \longrightarrow$ Law of $(\bar\sigma,\theta)$

$\theta \sim \mathrm{Unif}(\{\pm 1\})$, $G \sim N(0,1)$ $\boxed{\bar\sigma} \sim D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))$

(B) $$s_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}[\psi(\langle\sigma_i\rangle_{\beta,\lambda}, \theta_i)]/n$$

Need more work to show this.

$$= \mathbb{E}_{G,\theta}[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta)].$$

Interpretation: $\frac{1}{n}\sum_{i=1}^n \delta_{(\langle\sigma_i\rangle_{\beta,\lambda},\theta_i)} \to$ Law of (m,θ)

$\theta \sim \mathrm{Unif}(\{\pm 1\})$, $G \sim N(0,1)$. $\boxed{m} = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)$.

$m = \mathbb{E}[\bar\sigma]$.

Example 1: $\varphi(a, \theta) = (a - \theta)^2$.

$$(B) \Rightarrow \lim_{n\to\infty} \mathbb{E}[\| \langle \sigma \rangle_{\beta,\lambda} - \theta \|_2^2]/n = \mathbb{E}_{G,\theta}\left[\left(\tanh(2\beta\lambda\mu_\star\theta + 2\beta\sqrt{q_\star}G) - \theta\right)^2\right]$$

Example 2: $\varphi(a, \theta) = (\mathrm{sign}(a) - \theta)^2$

$$(B) \Rightarrow \lim_{n\to\infty} \mathbb{E}[\| \mathrm{sign}(\langle \sigma \rangle_{\beta,\lambda}) - \theta \|_2^2]/n = \mathbb{E}_{G,\theta}[(\mathrm{sign}(\tanh(\cdot\cdot)) - \theta)^2].$$

② Free energy trick and replica trick for $m_\star(\beta,\lambda)$.

$\Omega = \mathbb{Z}_2^n$, $\nu_0 = \mathrm{Unif}$

perturbation.

$$H_{\lambda,h}(\sigma) = -\langle \sigma, W\sigma \rangle - \lambda \langle \sigma, \theta \rangle^2/n - h \sum_{i=1}^n \psi(\sigma_i, \theta_i).$$

$$Z_n(\beta,\lambda,h) = \int_\Omega \exp\{-\beta H_{\lambda,h}(\sigma)\} \nu_0(d\sigma).$$

$$\varphi(\beta,\lambda,h) = \lim_{n\to\infty} \frac{1}{n} \mathbb{E}[\log Z_n(\beta,\lambda,h)].$$

$$m_\star(\beta,\lambda) = \frac{1}{\beta} \partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}.$$

We will show:

$$\varphi(\beta,\lambda,h) = \mathrm{ext}_{\mu,q}\, u(q,\mu;\beta,\lambda,h)$$

$$u(q,\mu;\beta,\lambda,h) = -\beta\lambda\mu^2 + \beta^2(1-q)^2 + \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}G\sigma + \beta h \psi(\sigma,\theta)\right]\right\}$$

$$(G,\theta,\sigma) \sim N(0,1) \times \mathrm{Unif}(\{\pm1\}) \times \mathrm{Unif}(\{\pm1\})$$

a) $S(k,\beta,\lambda,h) \equiv \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[Z_n(\beta,\lambda,h)^k]$. The n limit.

b) $\varphi(\beta,\lambda,h) \equiv \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,h)$. The k limit.

c) $m_\star(\beta,\lambda) \equiv \frac{1}{\beta} \partial_h \varphi(\beta,\lambda,h)\Big|_{h=0}$ The h derivative.

a) The n limit. Same as before, except the entropy term.

$$\mathbb{E}[Z_n^k] = \mathbb{E}\left[\left(\int_\Omega \exp\{-\beta H_{\lambda,h}(\sigma)\}\nu_0(d\sigma)\right)^k\right].$$

$$= \mathbb{E}\left[\int_{\Omega^k} \exp\{-\beta \sum_{a=1}^k H_{\lambda,h}(\sigma^a)\} \nu_0(d\sigma^a)\right].$$

$$= \int_{(\Omega)^{\otimes k}} \exp\left\{\beta\left(\sum_{a=1}^k \lambda \frac{\langle \theta, \sigma^a \rangle^2}{n} + h \sum_{a=1}^k \sum_{i=1}^n \psi(\sigma_i^a, \theta_i)\right)\right\}.$$

$$\times \underbrace{\mathbb{E}\left[\exp\{\beta \sum_{a=1}^k \langle \sigma^a, W\sigma^a \rangle\}\right]}_{E} \prod_{a=1}^k \nu_0(d\sigma^a)$$

$$E = \exp\{ \beta^2 \sum_{ab=1}^{k} \langle \sigma^a, \sigma^b \rangle^2 / n \}$$ Same as Spiked GOE.

$$\mathbb{E}[Z_n(\beta, \lambda, h)^k]$$

$$= \int_{(\Omega)^k} \exp\Big\{ \beta \sum_{a=1}^{k} \lambda \frac{\langle \theta, \sigma^a \rangle^2}{n} + \beta^2 \sum_{ab=1}^{k} \langle \sigma^a, \sigma^b \rangle^2 / n$$
$$+ \beta h \sum_{a=1}^{k} \sum_{i=1}^{n} \psi(\sigma_i^a, \theta_i) \Big\} \prod_{a=1}^{k} \nu_0(d\sigma^a).$$

$$\Big(\text{Plug in } 1 = \int_{\Omega^k} \prod_{a=1}^{k} \delta(\langle u, \sigma^a \rangle - n q_{0a}) \prod_{ab=1}^{k} \delta(\langle \sigma^a, \sigma^b \rangle - n q_{ab}) \prod d q_{ab} \Big)$$

$$= \sup_{\substack{Q \succeq 0 \\ Q_{ii}=1 \\ (x_a)_{a \in [k]}}} \exp\Big\{ \beta \lambda n \sum_{a=1}^{k} q_{0a}^2 + \beta^2 n \sum_{ab=1}^{k} q_{ab}^2 \Big\} \times \text{Ent}$$

$$\text{Ent} \equiv \int_{(\Omega)^k} \prod_{a=1}^{k} \delta(\langle \theta, \sigma^a \rangle - n q_{0a}) \prod_{1 \le a < b \le k} \delta(\langle \sigma^a, \sigma^b \rangle - n q_{ab})$$
$$\times \exp\Big\{ \beta h \sum_{a=1}^{k} \sum_{i=1}^{n} \psi(\sigma_i^a, \theta_i) \Big\} \prod_{a=1}^{k} \nu_0(d\sigma^a)$$ Large deviation calculation

Exercise !

$$\frac{1}{n} \log \text{Ent} = \inf_{\Lambda \in \mathbb{R}^{(k+1)\times(k+1)}} \Big\{ \langle Q, \Lambda \rangle / 2 + \log \mathbb{E}_\sigma \Big[\exp\Big\{ -\sum_{a,b=0}^{k} \lambda_{ab} \sigma_a \sigma_b / 2 + \beta h \sum_{a=1}^{k} \psi(\sigma_a, \sigma_0) \Big\} \Big] \Big\}$$

where $(\sigma_a)_{0 \le a \le k} \sim \text{Unif}(\{\pm 1\})$

$$\Rightarrow S(k, \beta, \lambda, h) = \sup_{\substack{Q \succeq 0 \\ Q_{ii}=1}} \inf_{\Lambda} U(Q, \Lambda)$$

$$U(Q, \Lambda) = \beta \lambda \sum_{a=1}^{k} q_{0a}^2 + \beta^2 \sum_{ab=1}^{k} q_{ab}^2$$
$$+ \langle Q, \Lambda \rangle / 2 + \log \mathbb{E}\Big[\exp\Big\{ -\sum_{ab=0}^{k} \lambda_{ab} \sigma_a \sigma_b / 2 + \beta h \sum_{a=1}^{k} \psi(\sigma_a, \sigma_0) \Big\} \Big].$$

Difference : ① Additional ψ term.

② The expectation doesn't have an analytical form.

b). The k limit

$$\varphi(\beta,\lambda,h) = \lim_{k\to 0} \frac{1}{k} \sup_{\substack{Q\succeq 0 \\ Q_{ii}=1}} \inf_{\Lambda} U(Q, X, \Lambda, \rho).$$

Replica symmetric ansatz,

$$\Pi = \begin{bmatrix} 1 & 0 \\ 0 & \bar{\Pi} \end{bmatrix} \Rightarrow U(Q,\Lambda) = U(\Pi Q \Pi^T, \Pi \Lambda \Pi^T).$$

$$Q = \begin{bmatrix} 1 & \mu & \cdots & \mu \\ \mu & 1 & & q \\ \vdots & & \ddots & \\ \mu & q & & 1 \end{bmatrix} \qquad \begin{matrix} q_{0a} = \mu & 1\le a\le k \\ q_{ab} = q & 1\le a\ne b\le k. \end{matrix}$$

$$\Lambda = \begin{bmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_1 & \lambda_2 & & \lambda_3 \\ \vdots & & \ddots & \\ \lambda_1 & \lambda_3 & & \lambda_2 \end{bmatrix}$$

Take derivative w.r.t. q_{0a}, q_{ab}.

$$\lambda_{0a} = -2\beta\lambda q_{0a}, \quad 1\le a\le k,$$

$$\lambda_{ab} = -4\beta^2 q_{ab}, \quad 1\le a,b\le k.$$

$$U(Q) = -\beta\lambda \sum_{a=1}^{k} q_{0a}^2 - \beta^2 \sum_{ab=1}^{k} q_{ab}^2$$

$$+ \log \mathbb{E}\left[\exp\left\{2\beta\lambda \sum_{a=1}^{k} q_{0a}\sigma_a\sigma_0 + 2\beta^2 \sum_{ab=1}^{k} q_{ab}\sigma_a\sigma_b + \beta h \sum_{a=1}^{k} \psi(\sigma_a,\sigma_0)\right\}\right].$$

$$\left(\begin{cases} q_{0a} = \mu, & 1\le a\le k, \\ q_{ab} = q, & 1\le a\ne b\le k. \end{cases}\right)$$

$$= -\beta\lambda k\mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k$$

$$+ \log \mathbb{E}_\sigma\left[\exp\left\{\beta\lambda\mu \sum_{a=1}^{k}\sigma_a\sigma_0 + \underbrace{2\beta^2 q \sum_{ab=1}^{k}\sigma_a\sigma_b}_{} + \beta h \sum_{a=1}^{k}\psi(\sigma_a,\sigma_0)\right\}\right]$$

Problem : How to calculate $\lim_{k\to 0} \frac{1}{k} U(Q)$?

Trick 1: $G \sim N(0,1)$, $\mathbb{E}_G\left[e^{\lambda G \sum_{a=1}^k \sigma_a}\right] = \exp\left\{\frac{\lambda^2}{2}\left(\sum_{a=1}^k \sigma_a\right)^2\right\} = \exp\left\{\frac{\lambda^2}{2}\sum_{ab=1}^k \sigma_a\sigma_b\right\}$.

$$= -\beta\lambda k\mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k \qquad \left(\sigma = (\sigma_0, \cdots \sigma_k) \sim \mathrm{Unif}(\mathbb{Z}_2^{k+1})\right)$$

$$+ \log \mathbb{E}_{G,\sigma}\left[\exp\left\{2\beta\lambda\mu\sum_{a=1}^k \sigma_a\sigma_0 + 2\beta\sqrt{q}\,G\sum_{a=1}^k \sigma_a + \beta h\sum_{a=1}^k \psi(\sigma_a,\sigma_0)\right\}\right]$$

$$= -\beta\lambda k\mu^2 - \beta^2(k + k(k-1)q^2) + 2\beta^2(1-q)k \qquad \sigma, \theta \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}\{\pm 1\}$$

$$+ \log \mathbb{E}_{G,\theta}\left[\mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\,G\sigma + \beta h\psi(\sigma,\theta)\}\right]^k\right]$$

Trick 2: $\lim_{k\to 0} \frac{1}{k}\log \mathbb{E}_X[Z(X)^k] = \mathbb{E}_X[\log Z(X)]$.

$$\Rightarrow \quad \lim_{k\to 0}\frac{1}{k}U(\mu,q) = -\beta\lambda\mu^2 \underbrace{- \beta^2(1-q^2) + 2\beta^2(1-q)}_{= +\beta^2(1-q)^2}$$

$$+ \mathbb{E}_{G,\theta}\left\{\log \mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu\sigma\theta + 2\beta\sqrt{q}\,G\sigma + \beta h\psi(\sigma,\theta)\}\right]\right\}$$

$$\equiv u(\mu, q; \beta, \lambda, h).$$

$$\Rightarrow \quad \varphi(\beta,\lambda,h) \equiv \operatorname*{ext}_{\mu,q}\; u(\mu, q; \beta, \lambda, h)$$

(A) is derived.

c) The h derivative

$$\frac{1}{\beta}\partial_h \varphi(\beta,\lambda,h)\Big|_{h=0} = \frac{1}{\beta}\partial_h u(\mu,q;\beta,\lambda,h)\Big|(q=q_*, \mu=\mu_*, h=0)$$

$$= \mathbb{E}_{G,\theta}\left[\frac{\mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q_*}G\sigma\}\psi(\sigma,\theta)\right]}{\mathbb{E}_\sigma\left[\exp\{2\beta\lambda\mu_*\sigma\theta + 2\beta\sqrt{q_*}G\sigma\}\right]}\right]$$

$$= \mathbb{E}_{G,\theta}\left[\mathbb{E}_{\bar\sigma}\left[\psi(\sigma,\theta)\right]\right], \qquad \mathbb{E}[\bar\sigma] = \tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G).$$

$$(\mu_*, q_*) = \operatorname*{arg\,ext}_{\mu,q}\; u(\mu,q;\beta,\lambda,h)\Big|_{h=0}.$$

$$\Rightarrow \begin{cases} \mu_* = \mathbb{E}_{G,\theta}\left[\theta\cdot\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)\right]. \\ q_* = \mathbb{E}_{G,\theta}\left[\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G)^2\right]. \end{cases}$$

Example: $\psi(\sigma, \theta) = \sigma \cdot \theta$.

$$\mathbb{E}[\langle \hat{\theta}, \theta \rangle / n] = \mathbb{E}[\langle \sum_{i=1}^n \sigma_i \theta_i / n \rangle_{\beta,\lambda,h}]\Big|_{h=0}$$

$$\xrightarrow{n\to\infty} \mathbb{E}_{G,\theta}\Big[\mathbb{E}_{\sigma \sim D(\theta, G)} [\sigma\theta] \Big]$$

$$= \mathbb{E}_{G,\theta}\Big[\theta \cdot \tanh(2\beta\lambda\mu_* \theta + 2\beta\sqrt{q_*}G)\Big] = \mu_*.$$

④ The free energy trick and replica trick for $S_*(\beta, \lambda)$.

Observation: Let $(\sigma^1, \sigma^2, \cdots \sigma^N) \sim P_{\beta,\lambda}^{\otimes N}$.

$$\Rightarrow \quad \frac{1}{N}\sum_{a=1}^N \sigma_i^a \xrightarrow{N\to\infty} \langle \sigma_i \rangle_{\beta,\lambda} \quad \text{(Law of large numbers)}.$$

$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big) \xrightarrow{N\to\infty} \frac{1}{n}\sum_{i=1}^n \psi(\langle \sigma_i \rangle_{\beta,\lambda}, \theta_i)$$

Define $\langle f(\sigma_1, \cdots \sigma_N) \rangle_{\beta,\lambda,N} \equiv \int_{\Omega^{\otimes N}} f(\sigma_1, \cdots, \sigma_N) \prod_{a=1}^N P_{\beta,\lambda}(d\sigma)$.

$$S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=1}^n \psi(\langle \sigma_i \rangle_{\beta,\lambda}, \theta_i)\Big]/n = \lim_{n\to\infty}\lim_{N\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]$$

$$\overset{?}{=} \lim_{N\to\infty}\lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]$$

Free energy trick:

$$\underline{\Omega} = \Omega^{\otimes N}, \qquad \underline{\nu_0} = \nu_0^{\otimes N}, \qquad \underline{\sigma} = (\sigma^1, \sigma^2, \cdots \sigma^N).$$

$$H_{\lambda,h,N}(\underline{\sigma}) = \sum_{a=1}^N H_\lambda(\sigma^a) - h\sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)$$

$$\varphi(\beta, \lambda, h, N) = \lim_{n\to\infty} \frac{1}{n} \log \int_{\underline{\Omega}} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\} \underline{\nu_0}(d\underline{\sigma}) \quad ☆$$

$$\partial_h \varphi(\beta, \lambda, 0, N) = \lim_{n\to\infty} \mathbb{E}\Big[\Big\langle \sum_{i=1}^n \psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma_i^a, \theta_i\Big)\Big\rangle_{\beta,\lambda,N}\Big]/n.$$

We expect: $$S_*(\beta,\lambda) \equiv \lim_{n\to\infty} \mathbb{E}\Big[\sum_{i=1}^n \psi(\langle \sigma_i \rangle_{\beta,\lambda}, \theta_i)\Big]/n$$

$$= \lim_{N\to\infty} \partial_h \varphi(\beta, \lambda, 0, N).$$

In fact, we will show that

$$\partial_h \varphi(\beta, \lambda, 0, N) = \mathbb{E}_{G,\theta}\Big[\mathbb{E}_{\sigma^a \overset{i.i.d.}{\sim} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G))}\Big[\psi\Big(\frac{1}{N}\sum_{a=1}^N \sigma^a, \theta\Big)\Big]\Big]$$

so that $S_*(\beta,\lambda) = \mathbb{E}_{G,\theta}[\psi(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}G), \theta)]$.

Replica trick.

a). $S(k,\beta,\lambda,h,N) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\left(\int_{\underline{\Omega}} \exp\{-\beta H_{\lambda,h,N}(\underline{\sigma})\} \underline{\nu_0}(d\underline{\sigma})\right)^k$. the n limit.

b). $\varphi(\beta,\lambda,h,N) = \lim_{k\to 0} \frac{1}{k} S(k,\beta,\lambda,N)$. the k limit.

c). $S_*(\beta,\lambda) = \lim_{N\to\infty} \partial_h \varphi(\beta,\lambda,0,N)$. the h derivatives.

a) The n limit.

$$H_{\beta,\lambda,N}(\underline{\sigma}) = \sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^a) - h \sum_{i=1}^{n} \psi\left(\sum_{a=1}^{N} \sigma_i^a, \theta_i\right)$$

$\mathbb{E}[Z_n]^k$

$$= \mathbb{E}\left[\int_{\Omega^{\otimes Nk}} \exp\left\{-\beta \sum_{b=1}^{k}\left(\sum_{a=1}^{N} H_{\beta,\lambda}(\sigma^{ab}) + \beta h \sum_{i=1}^{n} \psi\left(\sum_{a=1}^{N}\sigma_i^{ab},\theta_i\right)\right)\right\} \prod_{a=1}^{N}\prod_{b=1}^{k} \nu_0(d\sigma^{ab})\right.$$

Two types of replicas, form k groups.

replicas introduced in the replica trick ↓			
	σ^{11} - - -	σ^{1N}	Group 1
	σ^{21} - - -	σ^{2N}	Group 2
	⋮		⋮
	σ^{k1} - - - - -	σ^{kN}	Group k.

⟶

replicas introduced in the free energy trick

If $h = 0$: all the replicas are symmetric.

If $h \neq 0$: symmetry is broken.

Invariant if exchange replicas in the same group ✓

Invariant if exchange groups ✓

Non-invariant if exchange replicas in different groups. ✗

$$S(k,\beta,\lambda,h,N) = \sup_{\substack{\vec{\mu},\, Q\geq 0 \\ Q_{ii}=1}} U(\vec{\mu}, Q)$$

$$U(\vec{\mu}, Q) = -\beta\lambda \sum_{b=1}^{N}\sum_{a=1}^{k} \mu_{ab}^2 - \beta^2 \sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k} q_{ab,a'b'}^2$$

$$+ \log \mathbb{E}_{\sigma\theta}\left[\exp\left\{2\beta\lambda \sum_{b=1}^{k}\sum_{a=1}^{N} \mu_{ab}\sigma^{ab}\theta + 2\beta^2 \sum_{bb'=1}^{k}\sum_{aa'=1}^{N} q_{ab,a'b'}\sigma^{ab}\sigma^{a'b'} + \beta h \sum_{b=1}^{k} \psi\left(\frac{1}{N}\sum_{a=1}^{N}\sigma_i^{ab},\theta\right)\right\}\right]$$

$$\sigma = (\sigma^{ab})_{\substack{1\le a\le N\\ 1\le b\le k}} \overset{\text{i.i.d.}}{\sim} \mathrm{Unif}(\{\pm 1\}), \quad \theta \sim \mathrm{Unif}(\{\pm 1\})$$

$$Q = \begin{bmatrix} q_{11,11} & \cdots & q_{11,Nk} \\ \vdots & & \\ q_{Nk,11} & \cdots & q_{Nk,Nk} \end{bmatrix} \qquad \vec{\mu} = (\mu_{11}, \cdots, \mu_{Nk}).$$

b) The $k \to 0$ limit.

Replica symmetric ansatz $\quad (k=3, N=2)$

$$Q = \begin{array}{c} \\ \\ \end{array} \begin{array}{cc} \left[\begin{array}{cc|cc|cc} 1 & q_1 & q_2 & q_2 & q_2 & q_2 \\ q_1 & 1 & q_2 & q_2 & q_2 & q_2 \\ \hline q_2 & q_2 & 1 & q_1 & q_2 & q_2 \\ q_2 & q_2 & q_1 & 1 & q_2 & q_2 \\ \hline q_2 & q_2 & q_2 & q_2 & 1 & q_1 \\ q_2 & q_2 & q_2 & q_2 & q_1 & 1 \end{array}\right] & \begin{array}{l} \text{Group 1} \\ \\ \text{Group 2} \\ \\ \text{Group 3} \\ \end{array} \\ \begin{array}{ccc} \text{Group 1} & \text{Group 2} & \text{Group 3} \end{array} & \end{array}$$

$$\vec{\mu} = (\mu, \mu, \mu, \cdots \mu)$$

$$U(k, q_1, q_2, \mu, N) = -\beta\lambda Nk - \beta^2\left[Nk(1-q_1)^2 + N^2k(q_1-q_2)^2 + N^2k^2q_2^2\right]$$

$$+\log \mathbb{E}_{\sigma,\theta}\Big[\exp\Big\{2\beta\lambda\mu\sum_{ab}\sigma^{ab}\theta + 2\beta^2(1-q_1)Nk + 2\beta^2(q_1-q_2)\sum_{b=1}^{k}\sum_{a,a'=1}^{N}\sigma^{a'b}\sigma^{ab}$$

$$+ 2\beta^2 q_2 \sum_{b,b'=1}^{N}\sum_{a,a'=1}^{k}\sigma^{ab}\sigma^{a'b'} + \beta h\sum_{b=1}^{k}\psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\Big)\Big\}\Big]$$

$$= -\beta\lambda Nk - \beta^2\left[Nk(1-q_1)^2 + N^2k(q_1-q_2)^2 + N^2k^2q_2^2 - 2(1-q_1)Nk\right]$$

$$+\log \mathbb{E}_{\sigma,G_1,\theta}\Big[\exp\Big\{2\beta\lambda\mu\sum_{ab}\sigma^{ab}\theta + 2\beta^2(q_1-q_2)\sum_{b=1}^{k}\sum_{a,a'=1}^{N}\sigma^{a'b}\sigma^{ab}$$

$$+ 2\beta\sqrt{q_2}\,G_1\sum_{b=1}^{k}\sum_{a=1}^{N}\sigma^{ab} + \beta h\sum_{b=1}^{k}\psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^{ab}, \theta\Big)\Big\}\Big]$$

$$= -\beta\lambda Nk - \beta^2\left[Nk(1-q_1)^2 + N^2k(q_1-q_2)^2 + N^2k^2q_2^2 - 2(1-q_1)Nk\right]$$

$$+\log \mathbb{E}_{\theta,G_1}\Big[\Big(\mathbb{E}_\sigma \exp\Big\{2\beta\lambda\mu\sum_{a=1}^{N}\sigma^a\theta + 2\beta^2(q_1-q_2)\sum_{a,a'=1}^{N}\sigma^{a'}\sigma^{a}$$

$$+ 2\beta\sqrt{q_2}\,G_1\sum_{a=1}^{N}\sigma^a + \beta h\,\psi\Big(\frac{1}{N}\sum_{a=1}^{N}\sigma^a, \theta\Big)\Big\}\Big)^k\Big]$$

$$\varphi(\beta,\lambda,h,N) = \lim_{k\to 0} \sup_{q_1,q_2,\mu} \frac{1}{k} U(k,q_1,q_2,\mu,N) = \operatorname*{ext}_{\mu,q_1,q_2} u(q_1,q_2,\mu;\beta,\lambda,h,N)$$

$$u(q_1,q_2,\mu;\beta,\lambda,h,N) \equiv \lim_{k\to 0} \frac{1}{k} U(k,q_1,q_2,\mu,N)$$

$$= -\beta\lambda N - \beta^2\left[N(1-q_1)^2 + N^2(q_1-q_2)^2 - 2(1-q_1)N\right]$$
$$+ \mathbb{E}_{\theta,G_1}\Big[\log \mathbb{E}_\sigma\Big[\exp\Big\{2\beta\lambda\mu\sum_{a=1}^N \sigma^a\theta + 2\beta^2(q_1-q_2)\sum_{a,a'=1}^N \sigma^a\sigma^{a'}$$
$$+ 2\beta\sqrt{q_2}\,G_1\sum_{a=1}^N\sigma^a + \beta h\,\psi\Big(\frac{1}{N}\sum_{a=1}^N\sigma^a,\theta\Big)\Big]\Big]$$

$$= -\beta\lambda N - \beta^2\left[N(1-q_1)^2 + N^2(q_1-q_2)^2 - 2(1-q_1)N\right]$$
$$+ \mathbb{E}_{\theta,G_1}\Big[\log \mathbb{E}_{\sigma,G_2}\Big[\exp\Big\{2\beta\lambda\mu\sum_{a=1}^N \sigma^a\theta + 2\beta\sqrt{q_1-q_2}\,G_2\sum_{a=1}^N\sigma^a$$
$$+ 2\beta\sqrt{q_2}\,G_1\sum_{a=1}^N\sigma^a + \beta h\,\psi\Big(\frac{1}{N}\sum_{a=1}^N\sigma^a,\theta\Big)\Big]\Big].$$

c). $\partial_h \varphi(\beta,\lambda,h,N) = \partial_h u(q_1,q_2,\mu;\beta,\lambda,0,N)$

$$= \mathbb{E}_{\theta,G_1}\left[\frac{\mathbb{E}_{\sigma,G_2}\Big[\exp\Big\{2\beta\lambda\mu_*\sum_{a=1}^N\sigma^a\theta + 2\beta\big(\sqrt{q_{1*}-q_{2*}}\,G_2 + \sqrt{q_{2*}}\,G_1\big)\sum_{a=1}^N\sigma^a\Big\}\,\psi\Big(\frac{1}{N}\sum_{a=1}^N\sigma^a,\theta\Big)\Big]}{\mathbb{E}_{\sigma,G_2}\Big[\exp\Big\{2\beta\lambda\mu_*\sum_{a=1}^N\sigma^a\theta + 2\beta\big(\sqrt{q_{1*}-q_{2*}}\,G_2 + \sqrt{q_{2*}}\,G_1\big)\sum_{a=1}^N\sigma^a\Big\}\Big]}\right].$$

where $\mu_*, q_{1*}, q_{2*} = \operatorname*{arg\,ext}_{\mu,q_1,q_2} u(q_1,q_2,\mu;\beta,\lambda,0,N)$.

"Obviously", there exists a stationery point s.t.

$$q_{1*} = q_{2*} = q_*.$$

the $N=1$ formula

$$(\mu_*, q_*) = \operatorname*{arg\,ext}_{q,\mu} u(q,\mu;\beta,\lambda,0)$$

This is the correct stationery point for some region (β,λ).

The replica symmetric phase.

$$= \mathbb{E}_{\theta,G}\left[\mathbb{E}_{\sigma^a \overset{i.i.d.}{\sim} D(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G))}\Big[\psi\Big(\frac{1}{N}\sum_{a=1}^N\sigma^a,\theta\Big)\Big]\right]$$

$$\Rightarrow S_*(\beta,\lambda) = \lim_{N\to\infty}\partial_h\varphi(\beta,\lambda,0,N) = \mathbb{E}_{\theta,G}\left[\psi\big(\tanh(2\beta\lambda\mu_*\theta + 2\beta\sqrt{q_*}\,G),\theta\big)\right].$$

Remark: Sometimes even when $h=0$, there is spontaneous replica symmetry breaking.

The ansatz

$$Q = \begin{array}{cl} \left(\begin{array}{c|c|c} \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} & q_2 & q_2 \\ \hline q_2 & \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} & q_2 \\ \hline q_2 & q_2 & \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} \end{array}\right) & \begin{array}{l} \}\ \text{Group 1} \\ \}\ \text{Group 2} \\ \}\ \text{Group 3} \end{array} \\ \begin{array}{ccc} \underbrace{\qquad}_{\text{Group 1}} & \underbrace{\qquad}_{\text{Group 2}} & \underbrace{\qquad}_{\text{Group 3}} \end{array} & \end{array}$$

should be used when replica symmetric ansatz is not correct. This is called 1-step replica symmetry breaking. (1-RSB)

2-RSB ansatz.

$$Q = \left(\begin{array}{c|c} \begin{array}{c|c} \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} & q_2 \\ \hline q_2 & \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} \end{array} & q_3 \\ \hline q_3 & \begin{array}{c|c} \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} & q_2 \\ \hline q_2 & \begin{matrix}1 & q_1\\ q_1 & 1\end{matrix} \end{array} \end{array}\right)$$

The ground state of SK model when $\lambda=0$ is ∞-RSB.