Solving SDPs for synchronization and MaxCut problems via the Grothendieck inequality

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July 8, 2017

Joint work with Theodor Misiakiewicz, Andrea Montanari, and Roberto I. Oliveira

The MaxCut SDP problem

- $A \in \mathbb{R}^{n\times n}$ symmetric.
- MaxCut SDP:

$$\begin{aligned} \underset{X\in\mathbb{R}^{n\times n}}{\text{maximize}} \quad & \langle A, X\rangle \\ \text{subject to} \quad & X_{ii} = 1, \quad i \in [n], \\ & X \succeq 0. \end{aligned} \tag{SDP}$$

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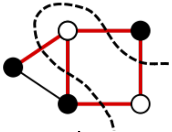

- G: a positively weighted graph. A_G: its adjacency matrix.
- MaxCut of G: known to be NP-hard

$$\underset{x\in\{\pm1\}^n}{\text{maximize}} \quad \frac{1}{4}\sum_{i,j=1}^{n} A_{G,ij}(1 - x_i x_j). \qquad \text{(MaxCut)}$$

- SDP relaxation: 0.878-approximate guarantee [Goemanns and Williamson, 1995]

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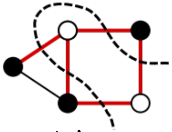

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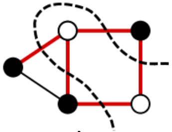

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- Convex formulation: n up to 10^3 using interior point method

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- Change variable $X = \sigma \cdot \sigma^{\mathsf{T}}$, $\sigma \in \mathbb{R}^{n\times k}$, $k \ll n$.
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- As $k \geq \sqrt{2n}$, the global maxima of the Non Convex formulation coincide with the global maximizer of the Convex formulation [Pataki, 1998], [Barviok, 2001], [Burer and Monteiro, 2003].

- As $k \geq \sqrt{2n}$, Non Convex formulation has no spurious local maxima [Boumal, *et al.*, 2016].

- What if k remains of order 1, as $n \to \infty$? Is there spurious local maxima?

- How is these local maxima?

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- What if k remains of order 1, as $n \to \infty$? Is there spurious local maxima? Sadly, yes.

- How is these local maxima? Empirically, they are good!

Geometry

$$\left.\begin{aligned} &\underset{\sigma \in \mathbb{R}^{n\times k}}{\text{maximize}} \quad \langle \sigma, A\sigma \rangle &&:= f(\sigma) \\ &\text{subject to} \quad \|\sigma_i\|_2 = 1. && \end{aligned}\right\} \mathcal{M}_k = \{\sigma \in \mathbb{R}^{n\times k} : \|\sigma_i\|_2 = 1\}.$$

Definition (ε-approximate concave point)

We call $\sigma \in \mathcal{M}_k$ an ε-approximate concave point of f on $\mathcal{M}_k$, if for any tangent vector $u \in T_\sigma \mathcal{M}_k$, we have

$$\langle u, \mathrm{Hess} f(\sigma)[u] \rangle \le \varepsilon \langle u, u \rangle. \tag{1}$$

Remark

A local maximizer is 0-approximate concave. An ε-approximate concave point is nearly locally optimal.

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Landscape Theorem

Theorem (A Grothendieck-type inequality)

For any ε-approximate concave point $\sigma \in \mathcal{M}_k$ of the rank-k non-convex problem, we have

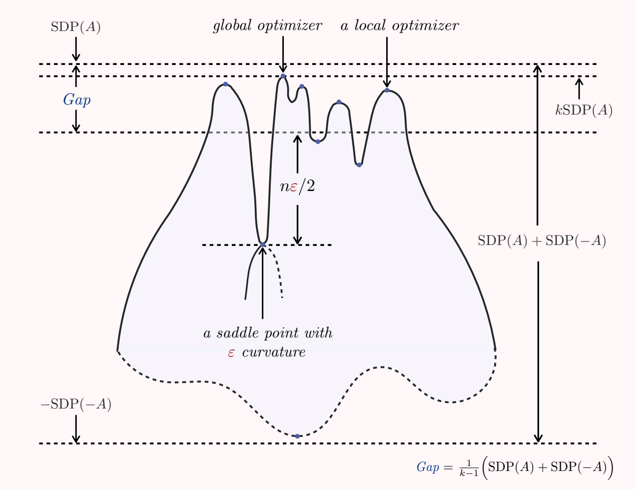
$$f(\sigma) \geq \mathrm{SDP}(A) - \frac{1}{k-1}(\mathrm{SDP}(A) + \mathrm{SDP}(-A)) - \frac{n}{2}\varepsilon. \tag{2}$$

SDP(A): the maximum value of SDP with input matrix A.

Geometric iterpretation: the function value for all local maxima are within a gap of order $O(1/k)$ within the global maximum.

Landscape of non-convex SDP

- $f(\sigma) \geq \mathrm{SDP}(A) - \frac{1}{k-1}(\mathrm{SDP}(A) + \mathrm{SDP}(-A)) - \frac{n}{2}\varepsilon.$



Efficient Algorithm

- Guaranteed converge rate using Riemannian trust region method.
- Getting absolute error $n\varepsilon + (\mathrm{SDP}(A) + \mathrm{SDP}(-A))/(k-1)$ within $c \cdot n||A||_1^2/\varepsilon^2$ trust region steps.
- Empirically, gradient descent converges faster than what is guaranteed.

Approximate MaxCut Guarantee

Theorem (Approximate MaxCut Guarantee)

For any $k \geq 3$, if $\sigma^\star$ is a local maximizer of corresponding rank-k non-convex problem, then we can use $\sigma^\star$ to find a $0.878 \times (1 - 1/(k-1))$-approximate MaxCut.

The global maximizer: 0.878-approximate MaxCut.

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Group Synchronization

- SO(d) synchronization, Orthogonal Cut SDP

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- Similar guarantee.

Conclusion

- Non-convex MaxCut SDP: all local maxima are near global maxima.
- An alternate algorithm for approximating MaxCut.
- Conclusion generalizable to general SDP problem.

What I did not go into detail

- $\mathbb{Z}_2$ synchronization and $\mathrm{SO}(d)$ synchronization.
- The one page proof for the Grothendieck-type inequality.

Questions?

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