Stat 155 Fall 2009: Solutions to Homework 8

(was due Nov. 12, 2009)

1. The game is symmetric, therefore $A = B^T$. Now let
$$\tilde{K} = \{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in \Delta_m\} = \Delta_m \times \Delta_m.$$
Note that $\tilde{K}$ is closed, bounded and convex. We need to define a map $f : \tilde{K} \to \tilde{K}$ such that f is continuous, and then we can apply Brouwer's fixed point theorem. (And then, following the proof in the book, we need to show that this fixed point must be a Nash Equilibrium.) Define c_i as in the notes, and note that $d_i = c_i$ because of symmetry.
$$c_i = c_i(\mathbf{x}, \mathbf{x}) = \max\{A_{(i)}\mathbf{x} - \mathbf{x}^T A\mathbf{x}, 0\}$$
Note that since the game is symmetric, $A_{(i)}\mathbf{x} = \mathbf{x}^T B^{(i)}$, and c_i gives the gain (if any) of either player by switching from strategy $\mathbf{x}$ to pure strategy $\mathbf{e_i}$.
Now we can define our function f by $f(\mathbf{x}, \mathbf{x}) = (\mathbf{y}, \mathbf{y})$ where
$$y_i = \frac{x_i + c_i}{1 + \sum_{i=1}^m c_i}.$$
We see that y_i is clearly non-negative, and $\sum_{i=1}^m y_i = 1$, therefore $y_i \in \Delta_m$. Also f is continuous, since c_i is continuous.
By Brouwer's fixed point theorem, there exists a fixed point for f, say $\mathbf{p}$, with $f(\mathbf{p}, \mathbf{p}) = (\mathbf{p}, \mathbf{p})$. We need to show that p must be a Nash equilibrium.
Since $p_i = \dfrac{p_i + c_i}{1 + \sum_{i=1}^m c_i}$, $\implies p_i \sum c_i = c_i$. This gives us that $c_i(\mathbf{p}, \mathbf{p}) = 0$ for each i. Therefore, $A_{(i)}\mathbf{p} \le \mathbf{p}A\mathbf{p}$ for each i, which implies that for every $x \in \Delta_m$,
$$\mathbf{x}A\mathbf{p} \le \mathbf{p}A\mathbf{p}.$$
Since $A_{(i)}\mathbf{p} = \mathbf{p}^T B^{(i)}$, we see that $\mathbf{p}^T B^{(i)}\mathbf{x} \le \mathbf{p}A\mathbf{p}$ as well. □

2. Let the drivers be $x_1, \ldots, x_6$ and their associated costs be $c_1, \ldots, c_6$. Then it is clear that $c_1 = 19$, since k will increment by 1. Now x_2 will choose to use the other possible route to D, and thus c_2 will also be 19. Proceeding in this way for each driver, we see that $c_3 = \min(25, 25) = 25$, $c_4 = 25$, $c_5 = c_6 = 31$, bringing the total cost to 150 units.
If a super highway is introduced along segment AC, then drivers 1,2,3 and 5 will go on this to reduce their cost, and drivers 4 and 6 will go along the segments $AB - BD$ to minimize their costs. This will bring the total cost to **102** units.

3. Let the pure strategies for player I be given by s_1 and s_2 where s_1 is the route $AD - DC$ and s_2 is the route $AB - BC$, and the pure strategies for player II be given by r_1 and r_2 where r_1 is the route $BC - CD$ and r_2 is the route $BA - AD$. This results in the following payoff matrix: (with payoff = - cost).

$$\begin{array}{cc} & \text{II} \\ \text{I} & \begin{array}{c|cc} & s_1 & s_2 \\ \hline s_1 & (-5,-5) & (-7,-8) \\ s_2 & (-5,-4) & (-7,-7) \end{array} \end{array}$$

The pure Nash equilibria are at (s_1, r_1) and (s_2, r_1)