Stat 155 Fall 2009: Solutions to Homework 6

(was due October 29, 2009)

1. Following a preferential ranking system with 4 being the most desirable outcome and 1 the least; if the homeowner is player I and the burglar is player II, we can list the rankings as shown below, where $a = (G, G), b = (G, NG), c = (NG, NG), d = (NG, G)$, where G = gun, and NG = no gun:

A		B
b	4	d
c	3	c
d	2	b
a	1	a

This results in the following payoff matrix:

$$\begin{array}{c} \\ \text{Gun} \\ \text{No gun} \end{array} \begin{array}{cc} \text{Gun} & \text{No gun} \\ \left(\begin{array}{c} (1,1) \\ (2,4) \end{array}\right. & \left.\begin{array}{c} (4,2) \\ (3,3) \end{array}\right) \end{array}$$

A slightly different preference (as a homeowner, you might prefer the situation (G, G) to (NG, G)) will give a different payoff matrix:

$$\begin{array}{c} \\ \text{Gun} \\ \text{No gun} \end{array} \begin{array}{cc} \text{Gun} & \text{No gun} \\ \left(\begin{array}{c} (2,2) \\ (1,4) \end{array}\right. & \left.\begin{array}{c} (4,1) \\ (3,3) \end{array}\right) \end{array}$$

As long as you list the preferences, and have a consistent matrix, it is okay.

2. The pure Nash equilibria are given by (CO,IW) and (IW, CO). To determine the mixed equilibria, suppose that player I plays CO with probability p, where $0 < p < 1$ and plays IW with probability $1 - p$. Then player II's expected payoffs for playing CO and IW are, respectively, $1 \cdot p + (-1) \cdot (1-p)$ and $2p + (-a) \cdot (1-p)$. We are looking for mixed equilibria so that player II puts positive probability on each of the actions, thus we get:

$$p - 1(1 - p) = 2p - a(1 - p)$$

Thus, we see that $p = (a - 1)/a$. Since the game is symmetric, we get the same strategy for player II, and the mixed equilibrium is given by $(\mathbf{p}, \mathbf{p})$, where $\mathbf{p} = ((a-1)/a, 1/a)$. Is it possible that $(\mathbf{p}, \mathbf{e})$ is a Nash equilibrium, where $\mathbf{e}$ is a pure strategy?

3. Payoff matrix:

$$\begin{array}{c} \\ W \\ D \end{array} \begin{array}{cc} W & D \\ \left(\begin{array}{c} (10-2, 10-2) \\ (10, 10-7) \end{array}\right. & \left.\begin{array}{c} (10-7, 10) \\ (0, 0) \end{array}\right) \end{array}$$

The pure Nash equilibria are given by (W,D) and (D,W). To find the mixed equilibria, note that the game is symmetric, and so the strategies for both students are going to be the same. Let us suppose that mixed equilibrium is given by $(\mathbf{p}, \mathbf{p}$, where $\mathbf{p} = (p, 1 - p)$ and $0 < p < 1$. By the usual reasoning, we solve $8p + 3(1 - p) = 10p$ to get $p = 3/5$.

4. Let p be the probability that player I chooses to advertise in the morning, and so $1-p$ is the probability that player I advertises in the evening. Similarly define q and r for players II and III respectively. If player III chooses to advertise in the morning, then we get the following payoff matrix for players I and II:

$$\begin{array}{c} \\ \text{I} \end{array} \begin{array}{c} \\ M \\ E \end{array} \overset{\text{II}}{\left(\begin{array}{cc} M & E \\ (0,0,0) & (0,300,0) \\ (300,0,0) & (0,0,200) \end{array} \right)}$$

And if player III advertises in the evening, the payoff matrix becomes:

$$\begin{array}{c} \\ \text{I} \end{array} \begin{array}{c} \\ M \\ E \end{array} \overset{\text{II}}{\left(\begin{array}{cc} M & E \\ (0,0,300) & (200,0,0) \\ (0,200,0) & (0,0,0) \end{array} \right)}$$

Now assume that $0 < p < 1$. Then the expected payoff for player II, if he chooses to advertise in the morning, is given by (recall that $r =$ probability that III advertises in the morning):

$$r(p \cdot 0 + (1-p) \cdot 0) + (1-r)(p \cdot 0 + (1-p) \cdot 200)$$

while the expected payoff, for player II if he chooses the evening, is

$$r(p \cdot 300 + (1-p) \cdot 0) + (1-r) \cdot 0$$

Equating these two, we get:

$$(1-r)(1-p)(200) = 300rp$$

For a symmetric Nash equilibrium, $p = q = r$, and using this to solve for p, we need to solve the equation $p^2 + 4p - 2$, and thus we get that $p = -2 \pm \sqrt{6}$. Taking the value that is in the interval $(0,1)$ we see that $p \sim 0.45$.

5. Yes. We will look at the problem for two-person games. The relation "is better than", as defined in class, defines a partial order on the outcomes. It is a transitive relation, so if the outcome (a,b) is better than the outcome (c,d) which is better than (e,f), then (a,b) must be better than (e,f). Recall that a Pareto optimal outcome is one that has the property that there does not exist an outcome that is "better", in the sense discussed in class. If a game has **no** Pareto optimal outcome, then for every outcome (a,b) there exists some outcome (a',b') such that (a',b') is better than (a,b). Then, using the transitivity, we can order the outcomes. Since the game is finite, this chain is finite, and there must be some outcome (c,d) at the end of this chain. But then (c,d) must be Pareto optimal, and we arrive at a contradiction.