Stat 155 Fall 2009: Solutions to Homework 3

(was due September 24, 2009)

1. This is a sum of two subtraction games. Using the notation from class, we can call the subtraction sets S_4 and S_5. Then we know that if the two piles have n and m chips, then $g_4(n) = n \bmod 5$ and $g_5(m) = m \bmod 6$. For (n, m) to be a **P**-position, we must have $g((n, m)) = g_4(n) \oplus g_5(m) = 0$. This is true if and only if $g_4(n) = g_5(m)$, that is: $n \bmod 5 = m \bmod 6$. (Also see Example 2.13.)

2.
   - The loops in the flower become stalks, and since $1 \oplus 1 \oplus 1 = 1$, the flower has SG value 2.
   - In the case of the little girl, fuse the two vertices in her head, since there are two edges, this circuit reduces to a single vertex, as does the circuit that forms her skirt (since it has 4 edges). Proceeding in this way, and using the Colon principle, we see that the SG value of the girl is 3.
   - Keep in mind that the ground is considered a single vertex, so the legs actually form a circuit with two vertices and two edges. Using the Fusion principle on this circuit and the circuit forming the dog's head, and the Colon principle on the branches, we see that the SG value of the dog is 2.
   - Finally, using the Colon principle on the branches of the tree, we get a SG value of 3 for the tree.

   Now, $2 \oplus 3 \oplus 2 \oplus 5 = 6$, and the easiest way to make this 0 is to hack off the first part of the upper branch of the bottom right branch of the tree. This will take off a value of 2, and we will be left with a value of 3, thus taking the SG value of the sum game to 0.

3. There is more than one way to show this. You could use part (a) and say that every tree has at least one leaf, so the tree on $n+1$ vertices must have at least 1 leaf. Removing the edge connecting this leaf to the rest of the tree will leave a single vertex and a tree on n vertices, and that must have $n-1$ edges by our assumption, so the original tree must have had n edges.
   Another possible proof is to show that if you have more than n edges on the tree on $n+1$ vertices, you will create a cycle. Start with the tree on 1 vertex, then assume that every tree on n vertices has $n-1$ edges. Now pick any tree on n vertices, and add a vertex. For this to be a tree on $n+1$ vertices, this vertex must have at least one edge connecting it to the n-vertex tree. Also, it cannot have more than 1 edge connecting it to another vertex in the n-vertex tree, since that would create two distinct paths between two vertices on the $n+1$-vertex graph, which would then no longer be a tree. (This proof would also need an argument that *every* tree on $n+1$ vertices can be obtained by adding a vertex and one edge to some tree on n vertices.)
   Here is another proof:
   Clearly, if we have a single vertex, then we have no edges and we have a tree on $n = 1$ vertices with 0 edges. Assume that every tree on k vertices has $k-1$ edges, for $k = 1, 2, \ldots, n$. Now consider a tree on $n+1$ edges. Remove an edge - this must disconnect the tree (since it has no cycles), into two smaller trees, of say l and m vertices each, where $l + m = n + 1$. Now, by our assumption, the tree with l vertices must have $l-1$ edges, and the tree with m vertices must have $m-1$ edges, and thus the original tree must have had $l - 1 + m - 1 + 1 = n + 1 - 1 = n$ edges.