Stat 155 Fall 2009: Solutions to the Practice Final, December 13, 2009
Duration: 3 hours

Please show **all** steps.

1. Solve the game of nim in which you are only allowed to take an **odd** number of beans from a heap. Is the position (5,45,3) in N or in P? If it is in N, what would be a winning first move?

   If you had only one heap, with an odd number of beans, you could take them all, so if n is the number of beans, n odd would be an N position. Thus, if n is even, n beans is a P position. The position $(5, 45, 3)$ will be in N since the nim-sum of its components is not zero. Any move that leaves a P position would be a winning first move, that is, the nim-sum of components should be 0.

2. Consider the following coin-turning game: there are a finite number of coins, each showing heads or tails. A move consists of turning over coins. Any number of coins may be turned over, but they must be consecutive, and the rightmost coin turned over must go from heads to tails. Find the Sprague-Grundy function for this game.

   This game ends when all the coins are T. So the position $TTT \dots T$ has SG value 0. If we have n coins, and number them from 1 to n, starting with the leftmost coin, then we can consider $g(x)$ to be the Sprague-Grundy value of the position where x is the rightmost coin that we will turn over. Then you can see that (check this for small n) $g(x) = \text{mex}\{0, g(n-1), g(n-1) \oplus g(n-2), \dots, g(n-1) \oplus g(n-2) \dots g(1)\}$. Working this out, you will see that $g(x)$ is the largest power of 2 dividing x. This is discussed in Ferguson, page I-31

3. Player II chooses a number $j \in \{1, 2, 3, 4\}$ and I tries to guess what it is. If the guess is correct, I wins a dollar. If the guess is too high, I loses a dollar, and if it is too low, there is no payoff. Set up the matrix of this game, and solve it.

   The payoff matrix is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

   resulting in an optimal strategy of $(\frac{8}{15}, \frac{4}{15}, \frac{2}{15}, \frac{1}{15})$ for player I, and a strategy of $(\frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15})$ for player II.
   The value of the game is $\frac{1}{15}$.

4. Find the optimal strategies for, and the value of the game whose matrix is given by:

$$\begin{pmatrix} 3 & -7 & 0 & -4 \\ -6 & 1 & -1 & 5 \end{pmatrix}$$

   The value of the game is $-13/3$, and the optimal strategies are given by $(2/3, 1/3)$, and $(0, 1/9, 0, 8/9)$.

5. State and prove Nash's theorem (you may assume Brouwer's fixed point theoerem).

   See the text for the proof.

6. Find all Nash equilibria in the general sum game given by:

$$\begin{pmatrix} (3,3) & (0,2) \\ (2,1) & (5,5) \end{pmatrix}$$

   The pure Nash equilibria are given by $\mathbf{p} = (0, 1) = \mathbf{q}$, $\mathbf{p} = (1, 0) = \mathbf{q}$. The mixed equilibrium strategy for I is given by $\mathbf{p} = (4/5, 1/5)$ and for II by $\mathbf{q} = (5/6, 1/6)$

7. State and prove Arrow's theorem.

   See the text for the proof.

8. Find the Shapley-Shubik index of the game that consists of 5 voters: A,B,C,D,E, who vote by majority rule, but A has a veto.

   The winning coalitions are all those that have A and at least 2 other voters. The Shapley-Shubik index is given by $\phi = (6/10, 1/10, 1/10, 1/10, 1/10)$.

9. Show that the Gale-Shapley algorithm for finding a stable matching yields a stable matching.

   See the text.

10. Consider the game with 5 sellers, where L_1, L_2, L_3 each have one lefthand glove, and R_1 and R_2 each have one righthand glove. The value of a coalition is the number of pairs of gloves it has. Find the core of this game, and the Shapley value. Is the Shapley value in the core?

    The core consists of a single imputation: $(0,0,0,1,1)$, and the Shapley value (which is not in the core is given by $\phi = (7/30, 7/30, 7/30, 13/30, 13/30)$.