Homework 1

Advanced Topics in Statistical Learning, Spring 2024

Due Friday February 9

1 Mathematical statistics warm-up [30 points]

(a) Suppose that $X_n \geq 0$ and $\mathbb{E}[X_n] = O(r_n)$. Prove that $X_n = O_p(r_n)$. [2 pts]

(b) Suppose that $X_n \geq 0$ and $X_n = O_p(r_n)$. Give an example to show that in general, this does not imply that $\mathbb{E}[X_n] = O(r_n)$. [2 pts]

(c) Prove that for $X \geq 0$, it holds that

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > t)\, dt.$$

You may assume that X is continuously distributed and hence has a probability density function. [4 pts]

(d) Suppose that $X_n \geq 0$ and $X_n = O_p(r_n)$, the latter bound holding "exponentially fast", meaning that there are constants $\gamma_0, n_0 > 0$ such that for all $\gamma \geq \gamma_0$ and $n \geq n_0$, we have

$$X_n \leq \gamma r_n, \quad \text{with probability at least } 1 - \exp(-\gamma).$$

Prove that $\mathbb{E}[X_n] = O(r_n)$. Hint: use the formulation for $\mathbb{E}[X_n]$ from the last question. [6 pts]

(e) Let $X_1, \ldots, X_n \sim P$, i.i.d., with $\mu = \mathbb{E}[X_i]$ and $\sigma^2 = \mathrm{Var}[X_i]$. Define

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \quad s_n^2 = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

(i) Prove that $s_n^2 \xrightarrow{p} \sigma^2$. [2 pts]

(ii) Prove that $\sqrt{n}(\bar{X}_n - \mu)/s_n \xrightarrow{d} N(0,1)$. [2 pts]

(f) Let $X \in \mathbb{R}^d$ and $Y \in \mathbb{R}$.

(i) Prove that $\mathbb{E}[(Y - f(X))^2]$ is minimized at $f(x) = \mathbb{E}[Y|X = x]$. [2 pts]

(ii) Prove that $\mathbb{E}[(Y - X^T\beta)^2]$ is minimized at $\beta = \Sigma^{-1}\alpha$, where $\Sigma = \mathbb{E}[XX^T]$ and $\alpha = \mathbb{E}[YX]$. [2 pts]

(g) This part will involve a small bit of coding. Attach your code in an appendix.

(i) Simulate Brownian motion on $[0,1]$, and a Brownian bridge on $[0,1]$, and plot them. [2 pts]

(ii) Simulate the 95th percentile of the supremum of the absolute Brownian bridge, i.e., the value q such that

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |B(t)| \geq q\Big) = 0.05.$$

where $B(t)$, $t \in [0,1]$ is the Brownian bridge. [2 pts]

(iii) Draw $X_1, \ldots, X_n \sim F$ from any distribution F of your liking (uniform, normal, etc.), calculate the Kolmogorov-Smirnov (KS) test statistic

$$T = \sqrt{n}\sup_x |F_n(x) - F(x)|,$$

where F_n is the empirical distribution of $X_1, \ldots, X_n$, and calculate the proportion of times out of (say) 1000 repetitions that T exceeds the threshold q computed in part (ii). [4 pts]

2 Risk analysis for least squares [30 points]

In this exercise, we will work on risk calculations for least squares regression.

(a) First, we start with an algebraic fact. Suppose that $A, B \succeq 0$, which we write to mean that are positive semidefinite matrices (symmetric with nonnegative eigenvalues). Prove that $\mathrm{tr}(AB) \geq 0$. [4 pts]

Hint: there are many ways to prove this, but for one, take an eigendecomposition of B, and expand the trace as a sum of products involving its eigenvectors.

(b) For this part and the next, suppose that we observe i.i.d. $(x_i, y_i) \in \mathbb{R}^d \times \mathbb{R}$. We write $f(x) = \mathbb{E}[y_i | x_i = x]$, $\epsilon_i = y_i - f(x_i)$, and assume that each $x_i \perp\!\!\!\perp \epsilon_i$. We denote $\sigma^2 = \mathrm{Var}[\epsilon_i]$.

Let $Y \in \mathbb{R}^n$ be the response vector and $X \in \mathbb{R}^{n \times d}$ the predictor matrix (whose i^{th} row is x_i). Let $\hat{\beta} = (X^\mathsf{T} X)^{-1} X^\mathsf{T} Y$ be the least squares solution of Y on X (where we assume that $X^T X$ is invertible, which requires $d \leq n$), and let $\hat{f}(x) = x^\mathsf{T} \hat{\beta}$.

Follow/reproduce the calculations in the review lecture to show that [6 pts]

$$\mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \mathrm{Var}(\hat{f}(x_i) \,|\, X)\right] = \sigma^2 \frac{d}{n},$$

and that, for an independent draw x_0 from the predictor distribution, [6 pts]

$$\mathbb{E}[\mathrm{Var}(\hat{f}(x_0) \,| X, x_0)] = \frac{\sigma^2}{n} \, \mathrm{tr}\left(\mathbb{E}[X^\mathsf{T} X] \, \mathbb{E}[(X^\mathsf{T} X)^{-1}]\right).$$

Therefore, using part (a), argue that [2 pts]

$$\mathbb{E}[\mathrm{Var}(\hat{f}(x_0) \,| X, x_0)] \geq \mathbb{E}\left[\frac{1}{n} \sum_{i=1}^{n} \mathrm{Var}(\hat{f}(x_i) \,|\, X)\right].$$

Hint: the calculations in lecture assumed the underlying model was linear and hence the bias (both in- and out-of-sample) was zero. But if you look back carefully, the variance calculations are unaffected by whether the true mean is linear or not.

(c) Follow/reproduce the calculations leading up to Theorem 1 in Rosset and Tibshirani (2020) to prove the inequality: [12 pts]

$$\mathbb{E}[\mathrm{Bias}^2(\hat{f}(x_0) \,| X, x_0)] \geq \mathbb{E}\left[\frac{1}{n} \sum_{\ell=1}^{n} \mathrm{Bias}^2(\hat{f}(x_i) \,|\, X)\right].$$

Note that you have shown that

$$\mathrm{Risk}(\hat{f}) \geq \mathbb{E}[\mathrm{Risk}(\hat{f}; x_{1:n})].$$

In other words, the out-of-sample risk of least squares is always at least as large as the in-sample risk (integrated over the feature values). To emphasize, this assumes nothing really at all (i.e., no underlying linear model) about the data model, except for the independence of x_i and ϵ_i.

(d) As a bonus, prove or disprove: there is a predictor distribution such that we get an equality in the last display, i.e., the out-of-sample and in-sample risks are equal. Note that we are still talking about standard least squares regression, so we are restricting attention to distributions such that $X^\mathsf{T} X$ is almost surely invertible.

3 Asymptotic scaling of nearest neighbor distances [32 points]

In this exercise, we will analyze the asymptotic scaling of nearest neighbor distances.

(a) Let $x_0, x_1, \ldots, x_n$ be i.i.d. from a distribution P supported on $[-R, R]^d$. Let $i(x_0)$ be the index of the closest point (in ℓ_2 distance) among $x_{1:n} = \{x_1, \ldots, x_n\}$ to x_0. Prove that for any $\delta > 0$, [6 pts]

$$\mathbb{P}(\|x_{i(x_0)} - x_0\|_2 > \delta) = \int (1 - P(B_d(x, \delta)))^n \, dP(x),$$

where $B_d(x, \delta)$ denotes the ℓ_2 ball of radius δ centered at x. To be clear, the probability on the left-hand side above is over x_0 and $x_{1:n}$.

(b) Prove that for any δ, there exists a rectangular partition $U_1, \ldots, U_{N(\delta)}$ of $[-R, R]^d$ with diameter at most δ, and [4 pts]

$$N(\delta) \le \frac{c}{\delta^d},$$

where $c > 0$ is a constant depending only on R and d.

(c) Using parts (a) and (b), prove that [12 pts]

$$\mathbb{P}(\|x_{i(x_0)} - x_0\|_2 > \delta) \le \frac{c}{en\delta^d}.$$

Hint: first show that

$$\mathbb{P}(\|x_{i(x_0)} - x_0\|_2 > \delta) \le \sum_{j=1}^{N(\delta)} \int_{U_j} (1 - P(U_j))^n \, dP(x) = \sum_{j=1}^{N(\delta)} P(U_j)(1 - P(U_j))^n.$$

Then show that each summand above is bounded by $1/(en)$.

(d) Argue that the last part translates to [2 pts]

$$\|x_{i(x_0)} - x_0\|_2 \lesssim \left(\frac{1}{n}\right)^{1/d} \quad \text{in probability.}$$

(e) Finally, let $k \ge 0$ be a nonnegative integer, and as in lecture, let $\mathcal{N}_k(x_0)$ denote the indices of the k closest (in ℓ_2 distance) points among $x_{1:n}$ to x_0. Use part (d) to prove that [8 pts]

$$\frac{1}{k} \sum_{i \in \mathcal{N}_k(x_0)} \|x_i - x_0\|_2 \lesssim \left(\frac{k}{n}\right)^{1/d} \quad \text{in probability.}$$

Hint: divide up the set $x_{1:n}$ into $k+1$ subsets, where the first k have equal size $\lfloor n/k \rfloor$. Let $i(x_0, j)$ denote the index of the closest point in subset j to x_0. Argue that

$$\sum_{i \in \mathcal{N}_k(x_0)} \|x_i - x_0\|_2 \le \sum_{j=1}^{k} \|x_{i(x_0,j)} - x_0\|_2,$$

and apply part (d) to each summand on the right-hand side.

4 Bonus: risk analysis for wavelet denoising

In this exercise, we will analyze the risk of wavelet denoising.

(a) Assume for now that we observe data according to the normal sequence model

$$z_\ell = \theta_\ell + \delta_\ell, \quad \ell = 1, \ldots, N, \tag{1}$$

where $\delta_\ell \sim N(0, \tau^2)$, independently, for $\ell = 1, \dots, N$. Consider the soft-thresholding estimator,

$$\hat{\theta}_\ell = S_\lambda(z_\ell) = \begin{cases} z_\ell - \lambda & \text{if } z_\ell > \lambda \\ 0 & \text{if } |z_\ell| \le \lambda \\ z_\ell + \lambda & \text{if } z_\ell < -\lambda \end{cases}, \quad \ell = 1, \dots, N.$$

Here $\lambda \ge 0$ is a tuning parameter. For arbitrary λ, prove that we have the exact risk expression:

$$\mathbb{E}\|\theta - \hat{\theta}\|_2^2 = \sum_{\ell=1}^N r(\theta_\ell, \lambda),$$

where

$$r(\mu, \lambda) = \mu^2 \int_{\frac{-\lambda-\mu}{\tau}}^{\frac{\lambda-\mu}{\tau}} \phi(z)\, dz + \int_{\frac{\lambda-\mu}{\tau}}^{\infty} (\tau z - \lambda)^2 \phi(z)\, dz + \int_{-\infty}^{\frac{-\lambda-\mu}{\tau}} (\tau z + \lambda)^2 \phi(z)\, dz,$$

and ϕ denotes the standard (univariate) normal density function.

(b) Prove that for $\lambda = \tau\sqrt{2 \log N}$, we have the risk upper bound:

$$\mathbb{E}\|\theta - \hat{\theta}\|_2^2 \le \tau^2 + (2 \log N + 1) \sum_{\ell=1}^N \min\{\theta_\ell^2, \tau^2\}.$$

Hint: start with $\tau^2 = 1$ for simplicity. Prove that, for any $\mu, \lambda \ge 0$, we have $0 \le \partial r(\mu, \lambda)/\partial \mu \le 2\mu$. From this, argue that $r(\mu, \lambda)$ is monotone increasing in μ, and further

$$r(\mu, \lambda) \le r(0, \lambda) + \min\{\mu^2, r(\infty, \lambda)\}.$$

Then, derive upper bounds on $r(0, \lambda)$ and $r(\infty, \lambda)$ (for the former you can use Mills' ratio, for the latter you can use direct arguments) to give

$$r(\mu, \lambda) \le e^{-\lambda^2/2} + \min\{\mu^2, 1 + \lambda^2\}.$$

Plug in $\lambda = \sqrt{2 \log N}$; show that an analogous bound holds for general $\tau^2 > 0$; and sum the bound over $\mu = \theta_\ell$, $\ell = 1, \dots, N$ to give the result.

(c) Now consider the nonparametric regression model

$$y_i = f(x_i) + \epsilon_i, \quad i = 1, \dots, n, \tag{2}$$

where $\epsilon_i \sim N(0, \sigma^2)$, independently, for $i = 1, \dots, n$, and $x_i \in [0, 1]$, $i = 1, \dots, n$ are fixed (more on them later). We are going to analyze the L^2 risk of a wavelet smoothing estimator $\hat{f}$,

$$\mathbb{E}\|f - \hat{f}\|_2^2 = \mathbb{E}\bigg[\int_0^1 (f(x) - \hat{f}(x))^2\, dx\bigg].$$

The estimator $\hat{f}$ will be defined by

$$\hat{f}(x) = \sum_{j,k} \tilde{\theta}_{jk}(y) \psi_{jk}, \tag{3}$$

where each ψ_{jk} is a Haar wavelet function, and each $\tilde{\theta}_{jk}(y)$ is a noisy empirical wavelet coefficient.

We begin with a simple Haar calculation. To recall the Haar basis on $[0, 1]$, first define $\psi(x) = 1\{x \in (0, 1/2]\} - 1\{x \in (1/2, 1]\}$. Then the Haar basis is given by the collection

$$1,\ \psi_{jk},\ \text{for } k = 0, \dots, 2^j - 1 \text{ and } j = 0, 1, 2, \dots,$$

where $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$. (For notational convenience, we let $\psi_{-10} = 1$, and implicitly index all basis calculations starting from $j = -1$.) Verify that this collection is orthonormal in L^2: show that the functions are pairwise orthogonal and unit norm, with respect to the L^2 inner product on $[0, 1]$,

$$\langle g, h \rangle = \int_0^1 g(x)h(x)\,dx.$$

(Accordingly the L^2 norm is simply given by $\|g\|_2^2 = \langle g, g \rangle = \int_0^1 g(x)^2\,dx$.)

(d) Explain why it is that we can write

$$\|f - \hat{f}\|_2^2 = \sum_{j,k} (\theta_{jk}(f) - \tilde{\theta}_{jk}(y))^2,$$

where the wavelet coefficients of f are

$$\theta_{jk}(f) = \langle f, \psi_{jk} \rangle = \int_0^1 f(x)\psi_{jk}(x)\,dx,$$

and $\tilde{\theta}_{jk}(y)$ are the coefficients to define the estimator $\hat{f}$ in its Haar basis expansion (3).

Hint: by orthonormality, observe that $f = \sum_{j,k} \theta_{jk}(f)\psi_{jk}$. It suffices to just name the theorem that relates the L^2 norm of a function to the norm of its coefficients.

(e) We define the last few parts needed to understand $\hat{f}$ and analyze its risk. For each j, k, we define the empirical wavelet coefficient

$$\tilde{\theta}_{jk}(f) = \frac{1}{n}\sum_{i=1}^n f(x_i)\psi_{jk}(x_i).$$

We also define a noisy empirical wavelet coefficient

$$\tilde{\theta}_{jk}(y) = \begin{cases} S_\lambda\left(\frac{1}{n}\sum_{i=1}^n y_i\psi_{jk}(x_i)\right) & \text{if } j \le j^* \\ 0 & \text{if } j > j^* \end{cases},$$

where S_λ is the soft-thresholding operator, as before, and j^* is a truncation level, to be chosen.

By part (d), and the inequality $(a + b)^2 \le 2a^2 + 2b^2$ (applied twice), we have

$$\mathbb{E}\|f - \hat{f}\|_2^2 \le 2 \underbrace{\sum_{j>j^*,k} \theta_{jk}^2(f)}_{e_1} + 4 \underbrace{\sum_{j\le j^*,k} (\theta_{jk}(f) - \tilde{\theta}_{jk}(f))^2}_{e_2} + 4 \underbrace{\mathbb{E}\left[\sum_{j\le j^*,k} (\tilde{\theta}_{jk}(f) - \tilde{\theta}_{jk}(y))^2\right]}_{e_3}.$$

We can interpret e_1 as the *truncation error*, e_2 as the *discretization error* (between population and empirical wavelet coefficients), and e_3 as the *estimation error* (in estimating the empirical wavelet coefficients from noisy data).

Denote by $\theta_{j\cdot}(f)$ the vector $(\theta_{jk}(f) : k = 0, \ldots, 2^j - 1)$. Assume that $\mathrm{TV}(f) \le 1$, and assume that the design points $x_i = i/n$, $i = 1, \ldots, n$ are evenly-spaced. It can be shown that

$$\|\theta_{j\cdot}(f)\|_1 \le c_1 2^{-j/2}, \quad \|\tilde{\theta}_{j\cdot}(f)\|_1 \le c_2 2^{-j/2}, \quad \text{and} \quad \|\theta_{j\cdot}(f) - \tilde{\theta}_{j\cdot}(f)\|_1 \le c_3 \frac{2^{j/2}}{n}, \tag{4}$$

for constants $c_1, c_2, c_3 > 0$. Use the first and third inequalities to show that there is a truncation level j^* such that sum of truncation and discretization errors satisfy $e_1 + e_2 \le C/n$, for another constant $C > 0$.

(f) It remains to study the estimation error. Assume that n is a power of 2. Show that, starting from the nonparametric regression model (2), we may transform this to a model of the form

$$z_\ell = \tilde{\theta}_\ell(f) + \delta_\ell, \quad \ell = 1, \dots, n,$$

where $\delta_\ell \sim N(0, \sigma^2/n)$, independently, for $\ell = 1, \dots, n$. Note that here, in indexing wavelet coefficients, we collapse the pair j, k into a single index ℓ.

Hint: use the appropriate truncation level j^*, from part (e), and only consider $j \le j^*$. Then define a matrix Ψ with elements $[\Psi]_{i\ell} = \psi_\ell(x_i)/n$, where in indexing the Haar wavelets, we again collapse the pair j, k into a single index ℓ. Using the fact we have an evenly-spaced design $x_i = i/n$, $i = 1, \dots, n$, show that $\Psi\Psi^\mathsf{T} = \frac{1}{n} I$, where I is the $n \times n$ identity matrix.

(g) Finally, note that from the transformation in part (f) you have brought yourself back to the problem studied in parts (a), (b): soft-thresholding under the sequence model (1), with noise level $\tau^2 = \sigma^2/n$.

From the risk bound from part (b), note that we have

$$\mathbb{E}\Bigg[\sum_{j \le j^*, k} (\tilde{\theta}_{jk}(f) - \tilde{\theta}_{jk}(y))^2\Bigg] \le \frac{\sigma^2}{n} + (2\log n + 1) \sum_{j \le j^*, k} \min\bigg\{\tilde{\theta}_{jk}^2(f), \frac{\sigma^2}{n}\bigg\}.$$

Use the second inequality in (4), on the empirical wavelet coefficients, to establish that for each j,

$$\sum_k \min\bigg\{\tilde{\theta}_{jk}^2(f), \frac{\sigma^2}{n}\bigg\} \le C\frac{\sigma^2}{n} 2^j \min\bigg\{1, 2^{-3j/2}\frac{\sqrt{n}}{\sigma}\bigg\},$$

for a constant $C > 0$. Show that gives the estimation error bound,

$$e_3 \le C \log n \bigg(\frac{\sigma^2}{n}\bigg)^{2/3}.$$

for a constant $C > 0$, redefined as needed.

Hint: the first bound (second-to-last display) is a bit tricky, whereas the second (last display) is more of a straight algebraic calculation, summing the first bound over j. To prove the first, argue that

$$\sup_{\|\tilde{\theta}_{j\cdot}\|_1 \le c_j} \sum_k \min\bigg\{\tilde{\theta}_{jk}^2, \frac{\sigma^2}{n}\bigg\}$$

will be achieved at a vector $\tilde{\theta}_{j\cdot}$ for which each entry is equal to 0 or $\sigma/\sqrt{n}$, except for (possibly) one entry, which is defined so that we hit the constraint $\|\tilde{\theta}_{j\cdot}\|_1 = c_j$. For the current problem, note that we have $c_j = c_2 2^{-j/2}$.

Concluding note: the risk bound you have shown, redefining the constant $C > 0$ as needed, is

$$\mathbb{E}\|f - \hat{f}\|_2^2 \le C\Bigg[\frac{1}{n} + \log n \bigg(\frac{\sigma^2}{n}\bigg)^{2/3}\Bigg],$$

for estimating a function with $\mathrm{TV}(f) \le 1$ using Haar wavelet denoising. This is minimax rate optimal for the class of functions with bounded TV, ignoring log factors (which could be removed from the upper bound with a slightly finer analysis).

References

Saharon Rosset and Ryan J. Tibshirani. From fixed-X to random-X regression: Bias-variance decompositions, covariance penalties, and prediction error estimation. *Journal of the American Statistical Association*, 15(529):138–151, 2020.