Statistics 200B
1997

Multivariate Normal Fact Sheet

A much more complete account can be found, for example, in C.R. Rao, *Linear Statistical Inference*. Here I just give some definitions and highlight some facts.

Definition: A random vector Y is said to follow a multivariate normal distribution with mean vector μ and covariance matrix Σ if for every vector a, $a^T Y$ follows a univariate normal distribution with mean $a^T\mu$ and variance $a^T\Sigma a$. We write, $Y \sim MVN(\mu, \Sigma)$.

Fact 1: If $Y \sim MVN(\mu, \Sigma)$ and A is a matrix compatible in dimension with Y, then $AY \sim MVN(A\mu, A\Sigma A^T)$

Fact 2: Two jointly multivariate normal vectors Y and Y are independent if and only if their cross covariance matrix is 0.

Fact 3: If Y and X are jointly multivariate normal, then the conditional distribution of Y given $X = x$ is $MVN(\mu_Y + \Sigma_{YX}\Sigma_{XX}^{-1}(x - \mu_X), \Sigma_{YY} - \Sigma_{YX}\Sigma_{XX}^{-1}\Sigma_{XY})$

Fact 4: If $Y \sim MVN(0, I)$ and P is a symmetric matrix of rank r such that $P^2 = P$, then $Y^T PY \sim \chi^2_r$.

Why? Suppose P is an $n \times n$ matrix. Consider an eigenvalue, λ, of P: $Px = \lambda x$. Then $\lambda x = Px = P^2 x = \lambda^2 x$ and since λ is real, λ must be 0 or 1. The eigenvalues of P thus consist of r ones and $n - r$ zeros. P can thus be expressed as $P = U\Lambda U^T$ where U is an orthogonal matrix ($U^T U = UU^T = I$) and Λ is a diagonal matrix of r ones and $n - r$ zeros. Now $Y^T PY = (U^T Y)^T\Lambda(U^T Y) = Z_1^2 + \cdots + Z_r^2$ where $Z = U^T Y$. Since $U^T U = I$, $\Sigma_{ZZ} = I$, so the Z_i are independent standard normal random variables. Thus $Z_1^2 + \cdots + Z_r^2 \sim \chi^2_r$

Corollary: If $Y \sim MVN(\mu, \sigma^2 I)$, and P is as above, then $\sigma^{-2}(Y - \mu)^T P(Y - \mu) \sim \chi^2_r$.

Fact 5: If the n-vector $Y \sim MVN(0, \Sigma)$, and Σ is invertible, then $Y^T\Sigma^{-1}Y \sim \chi^2_n$.

Why? Use the spectral decomposition of Σ: $\Sigma = U\Lambda U^T$, where Λ is a diagonal matrix consisting of the eigenvalues of Σ, which are positive since Σ is positive definite and U is an orthogonal matrix consisting of the eigenvectors. Then $\Sigma^{-1} = U\Lambda^{-1}U^T$ and $Y^T\Sigma^{-1}Y = (U^T Y)^T\Lambda^{-1}(U^T Y)$. Consider $Z = U^T Y$: it is MVN with mean 0 and covariance matrix $U\Sigma = \Lambda$. So the Z_i are independent mean 0 random variables with variances λ_i. Then

$$\begin{aligned} Y^T\Sigma^{-1}Y &= Z^T\Lambda^{-1}Z \\ &= \sum_{i=1}^{n} \frac{Z_i^2}{\lambda_i} \end{aligned}$$

Since the $Z_i/\sqrt{\lambda_i}$ are independent standard normal random variables, the distribution of the sum is χ^2_n.