Stat205B: Probability Theory (Spring 2003) **Lecture: 26**

Levy Process and Infinitely Divisible Law

Lecturer: James W. Pitman *Scribe: Bo Li* boli@stat.berkeley.edu

Levy Processes and Infinitely Divisible Law

Well begin the lecture with some classical distribution theory for Levy processes and infinitely divisible distributions.

Definition 1.1 *$X = \{X(t)\}_{t\geq 0}$ is said to be a Levy process if*

1. *X has independent increments.*
2. *$X(0) = 0$ a.s.*
3. *X is stochastically continuous (also called continuous in probability or $\mathbf{P}$-continuous), if, for $s \geq 0$*
$$X(t+s) - X(s) \xrightarrow{P} 0 \quad as \quad t \to 0$$
4. *X is time homogeneous, i.e., for $t \geq 0$, $\mathcal{L}(X(t+s) - X(s))$ does not depend on $s \geq 0$.*
5. *X is rcll (right continuous with left limits) almost surely.*

Without 5, X is said to be a Levy process in law.

Its easy to check that both Brownian Motion and Poisson process are Levy processes.

Theorem 1.1 *The marginal distribution of $X(t)$ is determined by $X(1)$.*

Proof: Since $X(t)$ has independent increments and is time homogenous, we can write

$$\begin{aligned}
\varphi_{s+t}(u) &= \mathbf{E}e^{iuX(s+t)} \\
&= \mathbf{E}e^{iu(X(s+t)-X(t))}e^{iuX(t)} \\
&= \mathbf{E}e^{iu(X(s+t)-X(t))}\mathbf{E}e^{iuX(t)} \\
&= \mathbf{E}e^{iuX(s)}\mathbf{E}e^{iuX(t)} \\
&= \varphi_s(u)\varphi_t(u)
\end{aligned}$$

The feature of $P-$continuity implies $\varphi_t(u)$ is continuous with respect to t for all u. Hammels theorem implies that $\varphi_t(u) = exp(t\phi(u))$, and further $\varphi_1(u) = exp(\phi(u))$. Hence $\varphi_t(u) = (\varphi_1(u))^t$. ■

Definition 1.2 *Two stochastic processes $\{Z_1(t)\}_{t\in T}$ and $\{Z_2(t)\}_{t\in T}$ are modifications (also called indistinguishable) of each other, if*

$$P\{Z_1(t) = Z_2(t)\} = 1 \quad for\ all \quad t \in T$$

There is a theorem that blurs the difference between Levy processes in law and Levy processes.

Theorem 1.2 *Each Levy process in law has a modification that is a Levy process.*

The general proof is delicate and well skip it.

Definition 1.3 *A random vector Y is infinitely divisible (id) if, for each $n \in \mathbb{N}$, there is an i.i.d. sequence $Y_{n,1}, \dots, Y_{n,n}$ so that*

$$Y \stackrel{d}{=} Y_{n,1} + \cdots + Y_{n,n}$$

The next result provides the basic link between Levy processes and triangular arrays.

Theorem 1.3 (Levy processes and infinitely divisibility) *For any random vector Y in $\mathbb{R}^d$, these conditions are equivalent:*

I. *Y is infinitely divisible.*

II. *$Y_{n,1} + \cdots + Y_{n,r_n} \xrightarrow{d} Y$ for some i.i.d. array $(Y_{n,j})_{n\geq 1, r_n \geq j \geq 1}$, where $r_n \to \infty$.*

III. *$Y \stackrel{d}{=} X_1$ for some Levy process X in $\mathbb{R}^d$.*

Two lemmas are needed for the proof.

Lemma 1.4 *If $Y_{n,j}$ are such as in II, then $Y_{n,1} \xrightarrow{P} 0$.*

Proof: Let μ and μ_n denote the distribution of Y and $Y_{n,1}$ respectively. Choose $r > 0$ so small that $\varphi_\mu \neq 0$ on $[-r, r]$, and write $|\varphi_\mu| = e^\psi$ on this interval, where $\psi : [-r, r] \to \mathbb{R}$ is continuous with $\psi(0) = 0$. For each $u \in [-r, r]$, since $\varphi_{\mu_n}^{r_n}(u) \to \varphi_\mu(u)$, we see for sufficiently large n, $\varphi_{\mu_n}(u) \neq 0$. Thus, we may write $|\varphi_{\mu_n}(u)| = e^{\psi_n(u)}$, where $r_n \psi_n(u) \to \psi(u)$. Hence $\psi_n(u) \to 0$ and therefore $\varphi_{\mu_n}(u) \to 1$. Now let $\epsilon \leq r^{-1}$, and note

$$\int_{-r}^{r} 1 - \varphi_n(u) du = 2r \int (1 - \frac{\sin rx}{rx}) \mu_n(dx) \geq 2r(1 - \frac{\sin r\epsilon}{r\epsilon}) \mu_n(|x| \geq \epsilon)$$

As $n \to \infty$, the left-hand side tends to 0 by dominated convergence theorem, and we get $\mu_n \xrightarrow{w} \delta_0$. ∎

Lemma 1.5 (Kolmogorov consistency) *Given distribution functions $\{\{F_t : \mathbb{R}^d \to [0, 1]\}_{t \in T^d}\}_{d \in \mathbb{N}}$, there exists a stochastic process $\{Z(t)\}_{t \in T}$ with these distributions as its Fidis, iff, the following two consistency conditions hold*

- $F_{\dots,t_{i-1},t_j,t_{i+1},\dots,t_{j-1},t_i,t_{j+1},\dots}(\dots, x_{i-1}, x_j, x_{i+1}, \dots, x_{j-1}, x_i, x_{j+1}, \dots) = F_t(x)$;
- $lim_{x_{k+1} \to \infty} F_{t,t_{k+1}}(x, x_{k+1}) = F_t(x)$.

Well skip the proof of this lemma.

Proof: *(of Theorem 1.3)* $I \Rightarrow II$ is trivial.
$II \Rightarrow I$. Write $m_n = [\frac{r_n}{2}]$ and let

$$S_{2n} = (Y_{2n,1} + \cdots + Y_{2n,m_n}) + (Y_{2n,n+1} + \cdots + Y_{2n,2m_n}) = Z_{n,1} + Z_{n,2}$$

The random vectors $Z_{n,1}$ and $Z_{n,2}$ are independent and have the same distribution. Lemma 1.4 implies $S_{2n} \Rightarrow Y$. Then the distribution of $Z_{n,1}$ is a tight sequence since

$$P(Z_{n,1} > z)^2 = P(Z_{n,1} > z)P(Z_{n,2} > z) \le P(S_{2n} > 2z)$$

and similarly $P(Z_{n,1} < -z)^2 \le P(S_{2n} < -2z)$. Similarly we see that $Z_{n,2}$ is also tight. So we can take a subsequence n_k so that $Z_{n_k,1} \Rightarrow Z_1$ and $Z_{n_k,2} \Rightarrow Z_2$. Then $Y \stackrel{d}{=} Z_1 + Z_2$. A similar argument shows that Y can be divided into $k \ge 2$ pieces.
$III \Rightarrow I$. $X(1) \stackrel{d}{=} X_{\frac{1}{n}} + (X_{\frac{2}{n}} - X_{\frac{1}{n}}) + \cdots + (X_1 - X_{\frac{n-1}{n}})$ yields the desired result.
$I \Rightarrow III$. We specify the law of $F_{t_1,\ldots,t_n}$ through that of $F_{t_1,t_2-t_1,\ldots,t_n-t_{n-1}}$, for $0 < t_1 < \ldots < t_n$, as that with ch.f. $\varphi_Y(\theta_1)^{t_1}\varphi_Y(\theta_n)^{t_2-t_1}\ldots\varphi_Y(\theta_n)^{t_n-t_{n-1}}$. These distributions are consistent. Thus Lemma 1.5 gives us a process X, with these Fidis, that must be a Levy process in law with $X(1) \stackrel{d}{=} Y$. By Theorem 1.1, these exists a Levy process with the same Fidis. ■

The following result is of the most fundamental importance in probability. The proof is not really difficult, but too technical to be worthwhile doing here.

Theorem 1.6 (Levy-Khintchine Formula) *Let X be a Levy process in $\mathbb{R}^d$. There exists a triplet (A,γ,ν) of*

$$\begin{cases} A & \text{a symmetric non-negative definite } d \times d \text{ matrix (the Gaussian convariance)} \\ \gamma & \text{a constant in } \mathbb{R}^d \\ \nu & \text{a measure on } \mathbb{R}^d \text{ with } \nu 0 = 0 \text{ and } \int_{\mathbb{R}^d}(|y|^2 \wedge 1)d\nu(y) < \infty \text{(the levy measure)} \end{cases}$$

which in that case is uniquely determined, such that, for all $u \in \mathbb{R}^d$ and $t \ge 0$, $\mathbf{E}e^{iuX(t)} = e^{t\psi_u}$, where

$$\psi_u = -\frac{1}{2} < u, Au > +i < u, \gamma > + \int_{\mathbb{R}^d} (e^{i<u,y>} - 1 - 1_{(|y|\le 1)} i < u, y >) d\nu(y)$$

If $\gamma_0 = \gamma - \int_{|y|\le 1} y d\nu(y)$ is well-defined and finite , then we may rewrite the Levy-Khintchine Formula with a new triplet $(A,\gamma_0,\nu)_0$ (the drift), as

$$\psi_u = -\frac{1}{2} < u, Au > +i < u, \gamma_0 > + \int_{\mathbb{R}^d} (e^{i<u,y>} - 1) d\nu(y)$$

If $\gamma_1 = \gamma + \int_{|y|>1} y d\nu(y)$ is well-defined and finite, then we can rewrite the Levy-Khintchine Formula with a new triplet $(A,\gamma_1,\nu)_1$ (the center), as

$$\psi_u = -\frac{1}{2} < u, Au > +i < u, \gamma_1 > + \int_{\mathbb{R}^d} (e^{i<u,y>} - 1 - i < u, y >) d\nu(y)$$

Definition 1.4 *A compound Poisson process is a Levy process with generating triplet $(0,0,\lambda\sigma)_0$, where $\lambda > 0$ is a constant and σ a probability measure on $\mathbb{R}^d$ with $\sigma\{0\} = 0$.*

Theorem 1.7 *Let $\{N(t)\}_{t\ge 0}$ be a Poisson process with rate λ, and $\{Y_k\}_{k=1}^{\infty}$ i.i.d. rv.s, independent of N, with $\mathcal{L}(Y_k) = \sigma$, where $\sigma\{0\} = 0$. Denoting $S_0 = 0, S_n = \Sigma_{k=1}^n Y_k$ for $n \in \mathbb{N}$, $X(t) = S_{N(t)}$ is a compound Poisson process with generating triplet $(0,0,\lambda\sigma)$.*

Proof: Rcll sample path and $X(0) \stackrel{d}{=} 0$ are immediate. $P-$continuity follows from

$$P\{|X(t+s) - X(s)| > \epsilon\} \leq P|N(s+t) - N(s) > 0\} = 1 - e^{-\lambda |t|} \to 0 \quad as \quad t \to 0$$

Independence and homogeneity of increments come by conditioning on the values of N involved. The generating triplet is the claimed one, since

$$\mathbf{E}e^{i<u,X(1)>} = \Sigma_{n=0}^{\infty} (\mathbf{E}e^{i<u,S_1>})^n \frac{\lambda^n}{n!} e^{-\lambda} = exp\{\lambda \int_{\mathbb{R}^d} (e^{i<u,y>} - 1) d\sigma(y)\}$$

■

Definition 1.5 *Y is stable if, for each $n \in \mathbb{N}$, with $Y_1, \ldots, Y_n$ i.i.d. copies of Y, $Y_1 + \ldots + Y_n \stackrel{d}{=} bY + c$ for some constants $b = b(n) > 0$ and $c = c(n) \in \mathbb{R}^d$. Y is strictly stable if it is possible to take $c(n) = 0$ for $n \in \mathbb{N}$.*

From the definition, its straightforward that stable rvs are id. Stable distributions are among the few most important id distributions. Two reasons are their stability under addition and the explicitness of their ch.f.(see below).

For an id random vector Y, Y^{*r} denotes an random vectores with ch.f. φ_Y^r.

Definition 1.6 *A stable Y is called $\alpha-$stable, $\alpha \in (0, 2]$, whenever*

$$Y^{*r} \stackrel{d}{=} r^{1/\alpha} Y + c \quad for\ t > 0\ and\ some\ constant\ c = c(r) \in \mathbb{R}^d.$$

Y is called strictly $\alpha-$stable if $c(t) = 0$ for $t > 0$.

For Y non-trivial (strictly) stable, there exists a unique constant $\alpha \in (0, 2]$ such that Y is $\alpha-$stable (strictly).

Definition 1.7 *A Levy process X with $X(1)$ (strictly) $\alpha-$stable is called a (strictly) $\alpha-$stable Levy motion.*

Theorem 1.8 *Let X be a non-trivial Levy process in $\mathbb{R}$ with generating triplet (A, γ, ν). Then X is $\alpha-$stable for some $\alpha > 0$ iff exactly one of these conditions holds:*

1. *$\alpha = 2$ and $\nu = 0$.*
2. *$\alpha \in (0, 2)$, $A = 0$, and $\nu(dx) = (c_+ 1_{(0,\infty)}(x) + c_- 1_{(-\infty,0)}(x))|x|^{-(\alpha+1)} dx$ on $\mathbb{R}$ for some $c_+, c_- \geq 0$.*

Proof: Suppose the generating triplet of $Y = X(1)$ is (A, γ, ν). We know Y is $\alpha-$stable iff $Y^{*r^\alpha} \stackrel{d}{=} rY + c$ for $t > 0$ and some constant c. Since the characteristics of Y^{*r^α} and $rY + c$ are $r^\alpha(A, \gamma, \nu)$ and $(r^2 A, r\gamma + c, \nu \circ S_r^{-1})$ respectively, where $S_r : x \mapsto rx$ for any $r > 0$. It follows that $X(t)$ is $\alpha-$stable iff $r^\alpha A = r^2 A$ and $r^\alpha \nu = \nu \circ S_r^{-1}$ for all $r > 0$. Thus $A = 0$ when $\alpha \neq 2$. Writing $F(x) = \nu[x, \infty)$ or $\nu(-\infty, x]$, then $r^\alpha \nu = \nu \circ S_r^{-1}$ implies $r^\alpha F(rx) = F(x)$ for all $r, x > 0$, and so $F(x) = x^{-\alpha} F(1)$. The condition $\int (x^2 \wedge 1) \nu(dx) < \infty$ implies $F(1) < \infty$ and when $\alpha = 2$ we have $F(1) = 0$. This complete the proof. ■

Reference:

[1]: Kallenberg, Foundation of Modern Probability

[2]: Patrik Albin, From Levy processes to Semimartingales (lecture notes) http://www.math.chalmers.se/ palbin/levy.html