Stat205B: Probability Theory (Spring 2003) **Lecture: 9**

Stationary Processes and the Invariant σ-field

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We first consider stationary processes, which will lead us to the basic setup for ergodic theory. Basic concepts are the invariant σ-field $\mathcal{I}$ and ergodicity. The intuitive meaning of these concepts becomes evident from examples.

Definition 9.1 (Stationary Process). $(X_0, X_1, X_2, \ldots)$ is said to be a **stationary process** if for $k = 1$, and hence for all $k = 1, 2, \ldots$,

$$(X_0, X_1, X_2, \ldots) \stackrel{d}{=} (X_k, X_{k+1}, X_{k+2}, \ldots)$$

That is, for each $n = 1, 2, \ldots$,

$$(X_0, \ldots, X_n) \stackrel{d}{=} (X_k, \ldots, X_{k+n})$$

In words, distribution of the sequence is invariant under shifts. Recall some examples we've seen before:

Example 9.2. A sequence $X_0, X_1, X_2, \ldots$ which is i.i.d., or more generally an exchangeable sequence.

Example 9.3. Markov chain (X_n) with stationary distribution π, started with X_0 distributed according to π.

For a more general construction of a stationary process, introduce

Definition 9.4. In probability space $(\Omega, \mathcal{F}, P)$, a measurable map $\varphi : \Omega \to \Omega$ is said to be **measure preserving** if $P(\varphi^{-1}A) = P(A)$ for any $A \in \mathcal{F}$.

Example 9.5 (general construction). We have a probability space $(\Omega, \mathcal{F}, P)$ and a measure preserving map φ. Let φ be the nth iterate of φ, φ^0 is identity map. Then for any $\mathcal{F}$-measurable function X, $X_n(\cdot) := X(\varphi^n(\cdot))$ defines a stationary process. To see this, notice for any $B \in \mathbb{R}^{n+1}$ and $A = \{\omega : (X_0(\omega), \ldots, X_n(\omega)) \in B\}$,

$$P((X_k, \ldots, X_{k+n}) \in B) = P(\varphi^k\omega \in A) = P(\omega \in A) = P((X_0, \ldots, X_n) \in B)$$

Remark. If (Y_n) is an arbitrary stationary process, then there exists (X_n) of the form just described, with $(X_n) \stackrel{d}{=} (Y_n)$. If (Y_n) has values in S, let P be the distribution induced by (Y_n) on the sequence space $S \times S \times \cdots, \mathcal{S} \times \mathcal{S} \times \cdots$, and let $X_n(\omega_0, \omega_1, \ldots) = \omega_n$. Let φ to be the shift operator, i.e. $\varphi(\omega_0, \omega_1, \ldots) = (\omega_1, \omega_2, \ldots)$ and $X(\omega) = \omega_0$. Then φ preserves P and $X_n(\omega) = X(\varphi^n\omega)$. This observation brings us to the basic setup for ergodic theory.

The basic setup for ergodic theory consists of

$(\Omega, \mathcal{F}, P)$	a probability space
φ	a P-preserving map
$X_n(\omega) = X(\varphi^n\omega)$	where X is an$\mathcal{F}$-measurable r.v.

The main subject of ergodic theory is the behavior of the averages

$$\frac{1}{n}\sum_{i=0}^{\infty} X_n$$

as $n \to \infty$.

Definition 9.6 (Invariant σ-field). A set $A \in \mathcal{F}$ is said to be **invariant** if $\varphi^{-1}A = A$. The collection of all such $A \in \mathcal{F}$ is the **invariant σ-field** $\mathcal{I}$.

Remark. By first considering indicators, one can easily show that a random variable X is $\mathcal{I}$-measurable iff $X = X \circ \varphi$. Then X is called **invariant**.

Notice that when φ is the shift on sequence space, the invariant σ-field $\mathcal{I}$ is contained in $\mathcal{T}$, the tail σ-field of $X_0, X_1, \ldots$. To see this, notice that if $A \in \mathcal{I}$, then by definition,

$$A = \{\omega : \omega \in A\} = \{\omega : \varphi\omega \in A\} \in \sigma\{X_1, X_2, \ldots\}$$

Iterating gives

$$A \in \cap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots) = \mathcal{T}$$

so $\mathcal{I} \subseteq \mathcal{T}$

Definition 9.7 (ergodicity). In the general setup, the P-preserving map φ is said to be **ergodic** if the invariant σ-field is trivial, that is $P(I) = 0$ or 1 for each $I \in \mathcal{I}$.

To decide in specific examples whether the invariant σ-field is trivial is not always easy. But here are some cases we can handle:

Example 9.8 (i.i.d. sequence). Here

$$\begin{aligned}
&\Omega = \mathbb{R} \times \mathbb{R} \times \cdots \\
&\mathcal{F} = \mathcal{B} \times \mathcal{B} \times \cdots \\
&P = \text{procuct measure} \\
&\varphi = \theta, \text{ shift operator} \\
&X_n(\omega) = \omega_n
\end{aligned}$$

By Kolmogorov's 0-1 law, the tail σ-field of the sequence is trivial, since $\mathcal{I} \subseteq \mathcal{T}$, the sequence is ergodic (i.e. under the setup, the shift is).

Sometimes the tail σ-field is not trivial, we have to check the ergodicity directly. See the following examples:

Example 9.9 (Markov chain). Here $(X_n, n \geq 0)$ is a irreducible Markov chain in countable state space S with a stationary distribution $\pi(x) > 0$, $\forall x$, which implies positive recurrence. Let $\varphi = \theta$, $\varphi^n = \theta_n$. Then if $A \in \mathcal{I}$, we have $\mathbf{1}_A = 1_A \circ \theta_n$, so let $h(x) = \mathbb{E}_x \mathbf{1}_A$ and $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$,

$$h(X_n) = \mathbb{E}_{X_n} \mathbf{1}_A = \mathbb{E}_\pi(\mathbf{1}_A \circ \theta_n | \mathcal{F}_n) = \mathbb{E}_\pi(\mathbf{1}_A | \mathcal{F}_n)$$

L*é*vy's 0-1 law says that the right-hand side converges to $\mathbf{1}_A$ as $n \to \infty$. While the recurrence says $\forall\, y \in S$, the left-hand side visits $h(y)$ i.o., so $h(\cdot)$ has to be a constant, and by the last sentence, $h \equiv 1$ or $h \equiv 0$, so $\mathbb{P}_\pi(A) = 1$ or $\mathbb{P}_\pi(A) = 0$. thus the invariant σ-field $\mathcal{I}$ is trivial.

Note that if all states have period $d > 1$, by previous work, we know $\mathcal{T} = \sigma(X_0 \in S_r : 0 \leq r < d)$, in which $S_0, S_1, \ldots, S_{d-1}$ is the cyclic decomposition of S. In this case the invariant σ-field $\mathcal{I}$ is trivial while tail σ-field $\mathcal{T}$ is not.

Example 9.10 (Rotation of the circle). Here $\Omega = [0, 1)$, $\mathcal{F} = \mathcal{B}$, $P =$ Lebesgue measure. Let $\theta \in (0, 1)$, define φ as $\varphi\omega = (\omega + n\theta) \bmod 1$. If we map $[0, 1)$ into $\mathbb{C}$ by $x \to exp(2\pi i x)$, we can see it's just the rotation of the circle.

The fact is, if we move by a rational multiple of 2π, i.e. $\theta \in \mathbb{Q}$, then φ is not ergodic; otherwise, φ is ergodic.

First we see the easy side, let $\theta = m/n < 1$, $m, n \in \mathbb{N}$, then for any $B \in \mathcal{B}$,

$$A = \cup_{k=0}^{n-1}(B + k/n)$$

is invariant. Take B to be a suitable interval to get an invariant set of probability p for any $p \in (0, 1)$.

To see the case when φ is irrational, notice a fact from Fourier analysis: if f is a measurable function on $[0, 1)$ with $\int f^2(x)dx < \infty$ then

$$\begin{aligned} f(x) &= \sum_k c_k e^{2\pi i k x} \\ &= \lim_{K\to\infty} \sum_{k=-K}^{K} c_k e^{2\pi i k x} \end{aligned}$$

which converges in the $\mathbf{L}^2[0, 1)$ sense. And the choice of c_k is unique

$$c_k = \int f(x)e^{-2\pi k x}dx$$

Now by definition of φ,

$$f(\varphi(x)) = \sum_k c_k e^{2\pi i k(x+\theta)} = \sum_k (c_k e^{2\pi i k\theta})e^{2\pi i k x}$$

The uniqueness of c_k implies $f(\varphi(x)) = f(x)$ iff

$$c_k(e^{2\pi i k\theta} - 1) = 0$$

When θ is irrational, we must have $c_k = 0$, $\forall\ k \neq 0$, which means f is constant.

Finally, for any $A \in \mathcal{I}$, apply the last fact to $f = \mathbf{1}_A$ shows $A = \emptyset$ or $[0, 1)$ a.s.