Stat205B: Probability Theory (Spring 2003) **Lecture: 6**

Proof of The Limit Theorem

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In this lecture, we will prove two limit theorem. Note here we consider Markov chains on a countable state space S.

Theorem 6.1. Suppose p is irreducible, aperiodic and has stationary distribution π. Then as $n \to \infty$, $p^n(x,y) \to \pi(y)$.

Proof. The idea is coupling. Let X_n, Y_n be independent copies of the Markov chain, while $X_0 = x$, $Y_0 \sim \pi$. Define $T = \inf\{n : X_n = Y_n\}$. First, lets suppose $T < \infty$ a.s., under this assumption, we have for any $k \le n$

$$\begin{aligned} P(X_n = y, T = k) &= \sum_{z \in S} P(X_n = y | X_k = z) P(X_k = z, T = k) \\ &= \sum_{z \in S} P(Y_n = y | Y_k = z) P(Y_k = z, T = k) \\ &= P(Y_n = y, T = k) \end{aligned}$$

So,

$$\begin{aligned} P(X_n = y) &= \sum_{k=0}^{\infty} P(X_n = y, T = k) \\ &= \sum_{k=0}^{n} P(Y_n = y, T = k) + P(X_n = y, T > n) \end{aligned}$$

The same for Y_n:

$$P(Y_n = y) = \sum_{k=0}^{n} P(Y_n = y, T = k) + P(Y_n = y, T > n)$$

Take substraction, we get

$$|P(X_n = y) - P(Y_n = y)| \le P(X_n = y, T > n) + P(Y_n = y, T > n)$$

Sum over y,

$$\sum_{y \in S} |P(X_n = y) - P(Y_n = y)| \le 2P(T > n) \to 0$$

which implies $p^n(x,y) \to \pi(y)$ for any y.

Now, we only need to show the assumption $T < \infty$ a.s. is true. Consider a new Markov chain as (X_n, Y_n) which is on the countable states space $S \times S$. Its transition probabilities are

$$\bar{p}((x_1, y_1), (x_2, y_2)) = p(x_1, x_2) p(y_1, y_2)$$

We will check two things: irreducibility and recurrence. Since p is irreducible and aperiodic, there exists K, L, s.t. $p^n(x_1, x_2) > 0$ for any $n > K$, $p^n(y_1, y_2) > 0$ for any $n > L$, so there exists N, for any $n > N$,

$\bar{p}^n((x_1, y_1), (x_2, y_2)) > 0$. As for the recurrence, it not hard to see that $\bar{\pi}(a, b) = \pi(a)\pi(b)$ defines a stationary distribution for $\bar{p}$.

The final observation is to notice T is the time for $\bar{p}$ hitting the diagonal, because $\bar{p}$ is irreducible and recurrent, we get $T < \infty$ a.s.. □

The second limit theorem may help explain the terms positive and null recurrent. Let

$$N_n(y) = \sum_{m=1}^{n} 1_{(X_m = y)}$$

be the number of visits to y by time n.

Theorem 6.2. Suppose y is recurrent. For any $x \in S$, as $n \to \infty$

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} 1_{(T_y < \infty)} \qquad P_x - a.s.$$

Proof. Suppose first we start at y. Let $R(k) = \min\{n \geq 1 : N_n(y) = k\}$ =the time of the kth return to y. Let $t_k = R(k) - R(k-1)$ where $R(0) = 0$. Since $X_0 = y$, $t_1, t_2, \ldots$ are i.i.d. and the strong law of large numbers implies

$$\frac{R(k)}{k} \to E_y T_y \quad a.s.$$

Since $R(N_n(y)) \leq n < R(N_n(y) + 1)$,

$$\frac{R(N_n(y))}{N_n(y)} \leq \frac{n}{N_n(y)} < \frac{R(N_n(y)) + 1}{N_n(y) + 1} \cdot \frac{N_n(y) + 1}{N_n(y)}$$

Let $n \to \infty$, and notice $N_n(y) \to \infty$ a.s. since y is recurrent, we have

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} \quad a.s.$$

When $x \neq y$, if $T_y = \infty$ then $N_n(y) = 0$ for all n, the result is true. On $\{T_y < \infty\}$, by strong Markov property, $t_2, t_3, \ldots$ are i.i.d. with $P_x(t_k = n) = P_y(T_y = n)$, so

$$\frac{R(k)}{k} = \frac{t_1}{k} + \frac{t_2 + \cdots + t_k}{k} \to E_y T_y \quad a.s.$$

With the same reasoning above, we can get

$$\frac{N_n(y)}{n} \to \frac{1}{E_y T_y} \quad a.s.$$

Combining this with the case for $\{T_y = \infty\}$ completes the proof. □

Remark. The theorem above says if we start from x then the asymptotic fraction of time spent at x is positive when x is positive recurrent and is 0 when x is null recurrent.

Reference: Section 5.5 of R. Durrett (1996). Probability: theory and examples.(2nd Edition) Duxbury Press.