Stat205B: Probability Theory (Spring 2003) **Lecture: 25**

Connections to Partial Differential Equations

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Recall Itô's formula for $f \in C^2$ and X a continuous semimartingale:

$$f(X) = F(X_0) + \sum_{i=1}^{n} \int_0^t f_{X_i}(X) dX^i + 1/2 \sum_{i,j=1}^{n} \int_0^t f_{X_i X_j}(X) d < X^i, X^j > \text{ a.s.}$$

For $X = B$, Brownian motion, we have

$$< B^i, B^j >_t = \delta_{ij} t$$

So

$$f(B) = f(B_0) + \sum_{i=1}^{n} \int_o^t f_{X_i}(B_s) dB_s + 1/2 \sum_{i,j=1}^{n} \int_o^t f_{X_i, X_j}(B_s) ds.$$

We begin by considering Laplace's equation:

$$\Delta u = 0.$$

Note that $\Delta u := \sum_{i=1}^{n} u_{Xi, X_i}$. C^2 functions with $\Delta u = 0$ are called *harmonic function*.

Let $B(x, r) := \{y : |y - x| < r\}$ and let D be an open subset of $\mathbb{R}^n$. Let $\tau_D := \inf\{t : B_t \in D^c\}$. Since each component of B is a.s. unbounded, $P(\tau_D < \infty) = 1$ for any bounded domain D.

Theorem 25.1. If u is harmonic in D, then

$$u(x) = ⨍_{\partial B(x,r)} u(y) dS$$

for every $x \in D$ and $r > 0$ such that $\overline{B(x, r)} \subset D$.

Proof. By Itô,

$$\begin{aligned} u(B_{t \wedge \tau_{B(x,r)}}) &= u(x) + \sum_{i=1}^{n} \int_0^{t \wedge \tau_{B(x,r)}} u_{X_i}(B_s) dB_s + 1/2 \int_0^{t \wedge \tau_{B(x,r)}} \Delta u(B_s) ds \\ &= u(x) + \sum_{i=1}^{n} \int_0^{t \wedge \tau_{B(x,r)}} u_{X_i}(B_s) dB_s \end{aligned}$$

Note that the second term in the last line is a local continuous martingale. But since $u(B_{t \wedge \tau_{B(x,r)}}) - u(x)$ is uniformly bounded, it is a true martingale with mean 0. Taking expectation and using $P(\tau_{B(x,r)} < \infty) = 1$,

$$\mathbb{E}_x \left(u(B_{\tau_{B(x,r)}}) \right) = u(X).$$

By the symmetry of Brownian motion we have

$$\mathbb{E}_x \left(u(B_{\tau_{B(x,r)}}) \right) = ⨍_{\partial B(x,r)} u(y) dS.$$

□

Theorem 25.2. If $u : D \to \mathbb{R}$ has the mean value property, then u is C^∞ and harmonic.

Now consider the equation

$$\Delta u = 0, \text{ in } D \text{ and } u = f \text{ on } \partial D \tag{1}$$

where $\mathbb{E}_x\left(|f(B_{\tau_D})|\right) < \infty$.

Claim. $u(x) := \mathbb{E}_x\left(f(B_{\tau_D})\right)$ has $\Delta u = 0$ in D.

Proof.

$$\begin{aligned} u(x) = \mathbb{E}_x f(B_{\tau_D}) &= \mathbb{E}_x\left(\mathbb{E}_x\left(f(B_{\tau_D})|\mathcal{F}_{\tau_{B(x,r)}}\right)\right) \\ &= \mathbb{E}_x u(B_{\tau_{B(x,r)}}) \quad \text{by strong Markov property} \\ &= \fint_{\partial B(x,r)} u(y)dS \end{aligned}$$

□

So in order to have a solution to the partial differential equation (1), we need:

$$\lim_{x\to a} \mathbb{E}_x\left(f(B_{\tau_D})\right) = f(a), \quad a \in \partial D.$$

This is true under a natural condition on the boundary. We want the boundary to be *regular*:

$$\text{if } \sigma_D = \inf\{t > 0 : B_t \in D^c\}, \text{ then } \mathbb{P}_x(\sigma_D = 0) = 1, \quad \forall x \in \partial D.$$

Stochastic Differential Equations

In order to consider more general PDEs, we need to introduce the notion of a *stochastic differential equations (SDEs)*. We say that the semimartingale X solves the SDE

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt$$

if

$$X_t = X_0 + \int_o^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds \tag{2}$$

Solutions to three equations exist in particular when σ and b are bounded and Lipschitz. The proof is based on Picard's iteration.

Claim. If X_t solves (2), then

$$M_t^f = f(X_t) - f(X_0) - \int_0^t Af(X)ds, \ t \geq 0, \ f \in C^2$$

where

$$Af(X) = 1/2 \sum_{i,j=1}^n a^{ij}(X) f_{X_i X_j}(X) + \sum_{i=1}^n b^i(X) f_{X_i}(X)$$

and $a = \sigma\sigma^T$, is a martingale.

Proof.

$$\begin{aligned} < X^i, X^j > &= \sum_{k,l=1}^{n} < \sigma^{ik}(X).B^k, \sigma^{jl}(X).B^l >_t \\ &= \sum_{k,l=1}^{n} \sigma^{ik}\sigma^{jl}. < B^k, B^l >_t \\ &= \int_0^t a^{ij} < X_s > ds. \end{aligned}$$

So by Itô's formula,

$$\begin{aligned} f(X_t) &= f(X_0) + \sum_{i=1}^{n} \int_0^t f_{X_i}(X) dX^i + 1/2 \sum_{i,j=1}^{n} \int_0^t f_{XiXj}(X_s) d < X^i, X^j >_s \\ &= f(X_0) + \sum_{i,j=1}^{n} \int_0^t \sigma^{ij}(X_s) f_{X_i}(X_s) dB_s^j + \int_0^t Af(X) ds \end{aligned}$$

Now assume that $u \in C^2(D) \cap C(\overline{D})$ is a solution of

$$-Au(X) = f(X) \text{ in } D, \text{ and } u = 0 \text{ on } \partial D.$$

Then $u(x) = \mathbb{E}_x \left(\int_0^{\tau_D} f(X_s) ds \right)$. By Itô,

$$\begin{aligned} u(X_{t\wedge\tau_D}) - u(x) &= M^f_{t\wedge\tau_D} + \int_0^{t\wedge\tau_D} Au(X_s) ds \\ &= M^f_{t\wedge\tau_D} - \int_0^{t\wedge\tau_D} f(X_s) ds \end{aligned}$$

Now taking expectation and limit as $t \to \infty$,

$$\mathbb{E}_x u(X_{\tau_D}) - u(x) = -\mathbb{E}_x \int_0^{\tau_D} f(X_s) ds$$

and so

$$u(x) = \mathbb{E}_x \int_0^{\tau_D} f(X_s) ds.$$

□

Dynamic Equations

We consider *dynamic equations* of the form

$$\begin{aligned} u_t &= Au - cu \ \text{ in } (0, \infty) \times \mathbb{R}^n \\ u(0, x) &= f(X) \end{aligned}$$

We show that the C^2 solutions of this equation are of the form

$$u(t, x) = \mathbb{E}_x \left(f(X_t) exp \int_0^t c(X_s) ds \right)$$

The first step is to show that $u(t-s, X_s)exp\left(-\int_0^t c(X_s)ds\right)$ is a local martingale on $[0, t)$.

If c, u is bounded, then M_s above is a bounded martingale. The martingale convergence theorem implies that as $s \nearrow t$, $M_s \to M_t$. Since u is continuous and $u(0, x) = f(x)$, we must have

$$lim_{s\nearrow t} M_s = f(B_t)exp\left(-\int_0^t c(X_s)ds\right).$$

So we have

$$\mathbb{E}_x f(X_t)exp\left(-\int_0^t c(X_s)ds\right) = u(t, x)$$

Applications:

We have seen that the solution to

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

is

$$u(x) = \mathbb{E}_x \int_0^{\tau_D} f(B_s)ds.$$

So if $f = 1$, we have $\mathbb{E}_x(\tau_D)$ is the solution of

$$\begin{aligned} -\Delta u &= f \text{ in } D \\ u &= 0 \text{ on } \partial D \end{aligned}$$

For example, if $D = B(0, 1)$, then the solution is $(1-|x|^2)/n$, $\Rightarrow$ $\mathbb{E}_x(\tau_{B(x,1)}) = (1-|x|^2)/n$.

References

[1] R. Bass: *Diffusions and Elliptic Operators*, Springer, New York (1998).

[2] O. Kallenberg: *Foundations of Modern Probability Theory*, Springer, New York (1997).

[3] I. Karatzas and S. Shreve: *Brownian Motion and Stochastic Calculus*(2nd ed.), Springer, New York (1991).