Stat205B: Probability Theory (Spring 2003) **Lecture: 24**

Quadratic Variation, Continued

Lecturer: Matthieu Cornec *Scribe: Brian Milch* `milch@cs.berkeley.edu`

Recall that in the last lecture we introduced the covariation process $[M, N]$ for local martingales M and N. We postponed the proof that this process exists. However, before we do that proof, lets prove one other proposition:

Proposition 24.1 *For any continuous local martingales M and N,*

$$[M - M_0, N - N_0] = [M, N] \text{ a.s.}$$

Proof: By the definition of $[M, N]$, $Z = MN - [M, N]$ is a local martingale. And by the definition of $[M - M_0, N - N_0]$, $W = (M - M_0)(N - N_0) - [M - M_0, N - N_0]$ is a local martingale. Therefore:

$$Z - W = M_0 N_0 + M_0(N - N_0) + N_0(M - M_0) - [M, N] + [M - M_0, N - N_0]$$

is also a local martingale. We can reduce to the case where $M_0 = N_0 = 0$, so $Z - W = -[M, N] + [M - M_0, N - N_0]$. Covariation processes are continuous, have finite variation, and start at 0, so by Prop. 23.1, $Z - W = 0$ a.s. Therefore $[M - M_0, N - N_0] = [M, N]$ a.s. ■

For convenience, here is Theorem 23.5 from last lecture. We will now do the existence portion of the proof.

Theorem 24.2 *For any continuous local martingales M and N, there exists an a.s. unique continuous process $[M, N]$, called the* **covariation process** *of M and N, such that $[M, N]$ has locally finite variation, $[M, N]_0 = 0$, and $MN - [M, N]$ is a local martingale.*

Proof: First, recall the basic equality:

$$uv = \frac{1}{4}\left((u + v)^2 - (u - v)^2\right)$$

So if the quadratic variations $[M + N]$ and $[M - N]$ exist, then we can define the covariation $[M, N]$ to be $\frac{1}{4}([M + N] - [M - N])$. This trick, called polarization, is generally useful for reducing a claim about products to a claim about squares.

So we just need to show $[M]$ exists with the desired properties. First, suppose M is a bounded, continuous martingale with $M_0 = 0$ a.s. For a fixed n, define a sequence (τ_k^n) of stopping times such that M_t changes by exactly 2^{-n} between τ_{k-1}^n and τ_k^n. That is:

$$\begin{aligned}
\tau_0^n &= 0 \\
\tau_k^n &= \inf\{t > \tau_{k-1}^n : |M_t - M_{\tau_{k-1}^n}| = 2^{-n}\}
\end{aligned}$$

Also define:

$$\begin{aligned}
V_t^n &= \sum_k M_{\tau_k^n} \mathbf{1}_{(\tau_k^n, \tau_{k+1}^n]}(t) \\
Q_t^n &= \sum_k \left(M_{\tau_{k+1}^n \wedge t} - M_{\tau_k^n \wedge t}\right)^2
\end{aligned}$$

Note that $(V_t^n)_{t\geq 0}$ is a predictable step process (Def. 23.4). Also:

$$M_t^2 = Q_t^n + 2(V^n \cdot M)_t$$

where $(V^n \cdot M)$ is a stochastic integral of a step process, as defined in the last lecture.

We claim that as $n \to \infty$, $(V^n \cdot M)$ converges in $\mathcal{M}^2$ to an L^2-bounded continuous martingale N. We will show this by showing that $(V^n \cdot M)$ is a Cauchy sequence in $\mathcal{M}^2$. First, note that V^n can be viewed as an approximation to M, where V^n takes the value $M_{\tau_k^n}$ over the whole interval $(\tau_k^n, \tau_{k+1}^n]$. Since M only changes by 2^{-n} over this interval, $|V^n - M| \leq 2^{-n}$.

To show that $(V^n \cdot M)$ is Cauchy, we must show that $\|V^n \cdot M - V^m \cdot M\|$ is small for large n, m. Integrating with respect to M is a linear operation, so:

$$\|V^n \cdot M - V^m \cdot M\| = \|(V^n - V^m) \cdot M\|$$

But:

$$\begin{aligned} |V^n - V^m| &\leq |V^n - M| + |V^m - M| \quad \text{by the triangle inequality} \\ &\leq 2^{-n} + 2^{-m} \\ &\leq 2^{-m+1} \quad \text{if } m < n \end{aligned}$$

So $\|V^n \cdot M - V^m \cdot M\| \leq 2^{-m+1}\|M\|$, and the sequence is clearly Cauchy.

Having shown $(V^n \cdot M) \to N$ as $n \to \infty$, we can define the quadratic variation as:

$$[M] = M^2 - N$$

Then $M^2 - [M] = N$, which is a martingale, as required by the theorem. We can view $[M]$ as the limit (as $n \to \infty$) of the Q_t^n. So intuitively, $[M]_t$ is the sum of the squared increments of M over infinitesimal sub-intervals covering the interval $[0, t]$.

For the case where M is not bounded, we use a standard localization argument. Define τ_n to be the first t such that $|M_t| > n$. Then M^{τ_n} is a bounded, continuous local martingale. So by the argument above, we can construct $[M^{\tau_n}]$ such that $(M^{\tau_n})^2 - [M^{\tau_n}]$ is a local martingale. Now let $\tau_n \uparrow \infty$. We can show that the $[M^{\tau_n}]$ are consistent on ts where they overlap, and thus define $[M]$ to equal M^{τ_n} for $t \leq \tau_n$ (this fully defines $[M]$ since $\tau_n \uparrow \infty$). Finally, we can show that $M^2 - [M]$ is a local martingale. ■

We will now prove a proposition that relates $[M]$ to the maximum value of M, which is $M^* = \sup_t |M_t|$.

Proposition 24.3 *For any sequence of continuous local martingales M_n starting at 0,*

$$M_n^* \xrightarrow{\mathbb{P}} 0 \ \Leftrightarrow\ [M_n]_\infty \xrightarrow{\mathbb{P}} 0$$

Proof: First suppose $M_n^* \xrightarrow{\mathbb{P}} 0$. For any fixed $\epsilon > 0$, let $\tau_n = \inf\{t : |M_n(t)| \geq \epsilon\}$. Let $N_n = M_n^2 - [M_n]$. Since N_n is a local martingale, $N_n^{\tau_n}$ is a true martingale; it also starts at 0. In particular, $E((N_n^{\tau_n})_{\tau_n}) = 0$, so:

$$E([M_n]_{\tau_n}) = E((M_n)_{\tau_n}^2) \leq \epsilon^2$$

Now by Chebyshevs inequality:

$$\begin{aligned} P([M_n]_\infty > \epsilon) &\leq \epsilon^{-1} E([M_n]_\infty) \\ &= \epsilon^{-1} E\left([M_n]_{\tau_n} \mathbf{1}_{\tau_n = \infty} + [M_n]_\infty \mathbf{1}_{\tau_n < \infty}\right) \\ &\leq \epsilon^{-1} E\left([M_n]_{\tau_n}\right) + \epsilon^{-1} E\left([M_n]_\infty \mathbf{1}_{\tau_n < \infty}\right) \\ &\leq \epsilon + \epsilon^{-1} E\left([M_n]_\infty \mathbf{1}_{\tau_n < \infty}\right) \quad \text{since } E([M_n]_{\tau_n}) \leq \epsilon^2 \end{aligned}$$

There is some way to show that $\epsilon^{-1} E\left([M_n]_\infty \mathbf{1}_{\tau_n < \infty}\right) \le P(\tau_n < \infty)$; Kallenberg does not provide this detail. But given that step,

$$\begin{aligned} P([M_n]_\infty > \epsilon) &\le \epsilon + P(\tau_n < \infty) \\ &= \epsilon + P(M_n^* > \epsilon) \\ &\to 2\epsilon \quad \text{as } n \to \infty, \text{ since } M_n^* \xrightarrow{\mathbb{P}} 0 \\ &\to 0 \quad \text{as } \epsilon \to 0 \end{aligned}$$

So we have $[M_n]_\infty \xrightarrow{\mathbb{P}} 0$. See the proof of Proposition 17.6 in Kallenberg for some hints on proving the converse. ∎