This lecture covers some of the technical background for the theory of stochastic integration. First, some notation: $M = (M_t)_{t \geq 0}$ is a process, and $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ is a filtration. We assume $\mathcal{F}$ is right-continuous and complete ($\mathcal{F}_t$ includes the null sets for each $t$). If $\tau$ is a stopping time, then $M^\tau$ is $M$ stopped at time $\tau$:

$$M^\tau = (M_{t \wedge \tau})_{t \geq 0}$$

### 23.1 Local martingales

**Definition 23.1** A process $M$ is a **local martingale** w.r.t. $\mathcal{F}$ if:

1. $M$ is adapted to $\mathcal{F}$, that is, $\forall t \quad M_t \in \mathcal{F}_t$
2. there exists a sequence $(\tau_n)$ of stopping times such that $\tau_n \uparrow \infty$ a.s., and $M^{\tau_n}$ is a true martingale for each $n$.

**Definition 23.2** $M$ is a **local $L^2$ martingale** if it satisfies Def. 23.1 with $M^{\tau_n}$ being an $L^2$ martingale for each $n$.

Other terms of the form “local <adjective> martingale” (e.g., local bounded martingale) are defined similarly: we require that each $M^{\tau_n}$ be an <adjective> martingale. Note that “<adjective> local martingale means something different: if we say that $M$ is a bounded local martingale, we are saying that $M$ is bounded and its a local martingale; were not saying anything special about the $M^{\tau_n}$.

**Remark 1:** If $M$ is a continuous local martingale, then we can take the $M^{\tau_n}$ to be bounded martingales. We can do this by letting $\tau_n = \inf\{t : |M_t| \geq n\}$; then since the paths are continuous, $|M^{\tau_n}| \leq n$.

**Remark 2:** Any continuous bounded local martingale is a true martingale. To see this, note that $M^{\tau_n} \uparrow M$, and since $M$ is bounded we can apply the dominated convergence theorem.

**Definition 23.3** Define the variation of $M$ over the interval $[0, t]$ as:

$$V_t(\omega) = \sup_{n \in \mathbb{N}} \sum_{i=1}^{n} |M_{t_i}(\omega) - M_{t_{i-1}}(\omega)|$$

Then $M$ has **locally finite variation** if $\forall t \exists C_t < \infty \quad V_t < C_t$ everywhere.
Local Martingales and Quadratic Variation

Proposition 23.1 (finite variation martingale) If $M$ is a continuous local martingale of locally finite variation then $M = M_0$ a.s.

Proof: We can reduce this to the case where $M$ is a true martingale with bounded variation and $M_0 = 0$ a.s. The reduction uses a localization argument: it suffices to show that $M^n = M_0$ a.s. for each $n$, and each $M^n$ is a true martingale. See the first paragraph of Kallenberg’s proof (p. 330) for details, and Figure 23.1 for intuition.

Now for a fixed $t$ and $n$, let $t_i = \frac{it}{n}$. Let:

$$
\xi_n \triangleq \sum_{i=1}^{n} (M_{t_i} - M_{t_{i-1}})^2 \\
\leq \left( \max_i |M_{t_i} - M_{t_{i-1}}| \right) V_t \\
\rightarrow 0 \text{ a.s. since } V_t \text{ is bounded}
$$

Note that $\xi_n \leq V_t^2$ for all $n$ and $EV_t^2 < \infty$, so the dominated convergence theorem implies $E\xi_n \to 0$ a.s. But:

$$
E\xi_n = E(M_t^2 - M_0^2) \quad \text{by orthogonality of martingale increments} \\
= EM_t^2 \quad \text{because } EM_0^2 = 0
$$

So $EM_t^2 = 0$, which implies $M_t = 0$ a.s.

We have proved this for arbitrary $t$, so we know $\forall t \ P(M_t = 0) = 1$. But we want to show $P(\forall t \ M_t = 0) = 1$. We do this by noting that $\forall t \ P(M_t = 0) = 1$ implies $P(\forall t \in Q^+ M_t = 0) = 1$, and then concluding that $P(\forall t \ M_t = 0) = 1$ because $M$ has continuous paths.

This proposition is used in proving the uniqueness of the covariation process (Thm. 23.5).

23.2 Stochastic integral of a step function

We want to define the stochastic integral $\int_0^t YdX$, where $(X_t)$ and $(Y_t)$ are both processes, and $\left( \int_0^t YdX \right)$ is another process. Kallenberg also uses the notation $(Y \cdot X)_t$ as a synonym for $\int_0^t YdX$.

The following definition handles the easy special case where $Y$ is a step process:
Definition 23.4 Suppose we have a process $X$, stopping times $\tau_k \uparrow \infty$, random variables $\eta_k \in \mathcal{F}_{\tau_k}$, and a predictable step process:

$$V_t = \sum_k \eta_k 1_{(\tau_k, \tau_{k+1}]}(t)$$

That is, $V_t$ equals $\eta_1$ on $(\tau_1, \tau_2]$, $\eta_2$ on $(\tau_2, \tau_3]$, etc. Then the stochastic integral of the step process $V$ with respect to $X$ is:

$$(V \cdot X)_t = \sum_k \eta_k (X^t_{\tau_{k+1}} - X^t_{\tau_k})$$  (23.1)

Recall that $X^t_{\tau_k}$ is the process $X$ stopped at time $t$, and evaluated at time $\tau_k$. So if $\tau_k > t$, then $X^t_{\tau_k} = X_t$.

One way to understand the stochastic integral is to imagine that $X^t_{\tau_{k+1}} - X^t_{\tau_k}$ is the fluctuation of a market between times $\tau_k$ and $\tau_{k+1}$, and $\eta_k$ is our “bet”, or the number of shares in the market that we own between $\tau_k$ and $\tau_{k+1}$. Then $\int_0^t VdX$ is the amount we gain in the market up to time $t$. By extending our definition of the stochastic integral to handle processes other than step processes, we will be able to model investment strategies that change the bet continuously.

Note that since $(V \cdot X)$ depends only on the changes in $X$:

$$(V \cdot X)_t = (V \cdot (X - X_0))_t$$

Recall that a martingale $M$ is in $L^2$ if $\sup_t EM_t^2 < \infty$.

Proposition 23.2 For any continuous $L^2$-martingale $M$ where $M_0 = 0$, and any predictable step process $V$ where $|V| \leq 1$, the process $(V \cdot M)$ is an $L^2$-martingale with $E(V \cdot M)_t^2 \leq EM_t^2$.

Proof: First assume there are only a finite number of nonzero terms in $V_t$. We use the following lemma from Chapter 7 of Kallenberg:

Lemma 23.3 If $M$ is a continuous martingale, $\tau$ is a stopping time, and $\zeta \in \mathcal{F}_\tau$, then the process $(N_t)_t = (\zeta(M_t - M_\tau))$ is also a martingale.

This process $N_t$ is zero up to time $\tau$, because for those times $M_t = M_\tau$. For $t \geq \tau$, $N_t = \zeta(M_t - M_\tau$.

We can rewrite the definition of $(V \cdot M)_t$ as a sum of processes of this form (see equation (1) in Chapter 17 of Kallenberg). So from the lemma and the assumption that the sum is finite, we can conclude that $(V \cdot M)_t$ is a martingale.

We still have to show $E(V \cdot M)_t^2 \leq EM_t^2$:

$$E(V \cdot M)_t^2 = E \left( \sum_k \eta_k (M^t_{\tau_{k+1}} - M^t_{\tau_k}) \right)^2$$

$$= E \left( \sum_k \eta_k^2 (M^t_{\tau_{k+1}} - M^t_{\tau_k})^2 \right) + 2E \left( \sum_{i < j} \eta_i \eta_j (M^t_{\tau_{i+1}} - M^t_{\tau_i})(M^t_{\tau_{j+1}} - M^t_{\tau_j}) \right)$$
But by the orthogonality of martingale increments, the second expectation is zero. So:

\[
E(V \cdot M)^2_t = E \left( \sum_k \eta_k^2 (M^t_{r_{k+1}} - M^t_{r_k})^2 \right) \\
\leq E \left( \sum_k (M^t_{r_{k+1}} - M^t_{r_k})^2 \right) \quad \text{since } |V| \leq 1 \\
= EM_t^2 \quad \text{by orthogonality of increments}
\]

For the general case where \( V \) has infinitely many nonzero terms, take \( V_j \to V \), where each \( V_j \) has finitely many nonzero terms. Then:

\[
E(V \cdot M)^2_t = E(\liminf_j (V_j \cdot M)^t) \quad \text{by Fatou's lemma} \\
\leq EM_t^2 \quad \text{by result proved above}
\]

This proves the second claim in the lemma, but we still need to show \( (V \cdot M)^t \) is a martingale. This is left as an exercise; the idea is to use dominated convergence.

\[\square\]

### 23.3 The space \( \mathcal{M}^2 \)

**Definition 23.5** For a fixed filtration \( \mathcal{F} \), define the space:

\[ \mathcal{M}^2 = \{ M : M \text{ is a continuous martingale with respect to } \mathcal{F} \text{ and is } L^2 \text{-bounded} \} \]

It can be shown that if \( M \in \mathcal{M}^2 \), then there is some \( M_\infty \) such that \( M_t \to M_\infty \) a.s. as \( t \to \infty \). The proof of this result builds on the fact that \( X_t \to X_\infty \) when \( X \) is a discrete \( L^2 \)-martingale; we then consider countable subsequences of the indices \( t \) for \( (M_t) \). Furthermore, given an \( M_\infty \), we can recover the process \( (M_t) \) such that \( M_t \to M_\infty \): by the definition of a continuous-time martingale, \( M_t = E(M_\infty | \mathcal{F}_t) \) for each \( t \).

Since each \( M \in \mathcal{M}^2 \) converges to some \( M_\infty \), we can define the norm:

\[
\|M\| = \|M_\infty\|_2 = (EM_\infty^2)^{1/2}
\]

**Proposition 23.4** For any \( M \in \mathcal{M}^2 \), let \( M^* = \sup_t |M_t| \). Then \( \|M^*\|_2 \leq 2\|M\| \).

**Proof:** Let \( \bar{M}_t = \sup_{s \in [0,t]} |M_s| \). By the \( L^2 \) maximum inequality (Thm. 4.4.3 of Durrett), for any \( t \):

\[
(E(\bar{M}_t)^2)^{1/2} \leq 2(EM_t^2)^{1/2} \leq 2\sup_t(EM_t^2)^{1/2}
\]

But \( M_t = E(M_\infty | \mathcal{F}_t) \), so because conditioning reduces variance:

\[
(E(\bar{M}_t)^2)^{1/2} \leq 2\sup_t(EM_\infty^2)^{1/2} = 2(EM_\infty^2)^{1/2} = 2\|M\|
\]

So \( \|\bar{M}_t\|_2 \leq 2\|M\| \) for each \( t \), which implies \( \|M^*\|_2 \leq 2\|M\| \).

\[\square\]

This proposition is used to prove that \( \mathcal{M}^2 \) is complete; see Lemma 17.4 in Kallenberg.
23.4 Covariation and quadratic variation

**Theorem 23.5** For any continuous local martingales $M$ and $N$, there exists an a.s. unique continuous process $[M, N]$, called the **covariation process** of $M$ and $N$, such that $[M, N]$ has locally finite variation, $[M, N]_0 = 0$, and $MN - [M, N]$ is a local martingale.

The existence portion of the proof will be done in the next lecture. Assuming such an $[M, N]$ exists, its uniqueness follows from Prop. 23.1. Also, given uniqueness, it is obvious that the form $[M, N]$ must be symmetric and bilinear.

**Definition 23.6** If $M$ is a continuous local martingale, the **quadratic variation** of $M$ is $[M, M]$.

**Proposition 23.6** For any continuous local martingales $M$ and $N$ and any stopping time $\tau$:

$$[M, N]^{\tau} = [M^{\tau}, N^{\tau}] = [M^{\tau}, N] \text{ a.s.}$$

**Proof:** The first inequality follows directly from the uniqueness of $[M, N]$. For the second, note that since $MN - [M, N]$ is a local martingale,

$$M^{\tau}N^{\tau} - [M, N]^{\tau}$$

is a local martingale. It can also be shown that whenever $N$ is a local martingale,

$$M^{\tau}(N - N^{\tau})$$

is a local martingale (Kallenberg cites his Corollary 7.14 for this fact). Adding the two local martingales together, we get another local martingale:

$$M^{\tau}N - [M, N]^{\tau}$$

But by Theorem 23.5, $M^{\tau}N - [M^{\tau}, N]$ is the unique local martingale obtained by subtracting a covariation process from $M^{\tau}N$, so it must be that $[M, N]^{\tau} = [M^{\tau}, N]$.

So what have we done so far? We were trying to understand functions of martingales, and were starting with polynomials — specifically, what is the product of two martingales $M$ and $N$? The product $MN$ is not generally a martingale, but Theorem 23.5 says that $MN - [M, N]$ is a local martingale. As an example of this, consider $B_t^2 - [B_t] = B_t^2 - t$, which we already knew was a martingale.